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ON THE ANALYTICAL FORM OF THE EARTH'S MAGNETIC
ATTRACTION EXPRESSED AS A FUNCTION OF TIME

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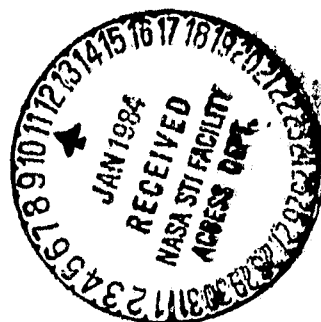
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16. Abstract An attempt is made to express the Earth's magnetic attraction in simple analytical form using observations during the 16th to 19th centuries. Observations of the magnetic inclination in the 16th and 17th centuries are discussed.			
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ON THE ANALYTICAL FORM OF THE EARTH'S MAGNETIC ATTRACTION
EXPRESSED AS A FUNCTION OF TIME

V. Carlheim-Gyllenskold**

A complete theory of the Earth's magnetic attraction
must include the force's manifestations for any given time. /3*

The old theory solves this problem for a specific period of time, but does not account for an aspect called secular variations. Generally, the variations of elements observed in a given place are expressed by empirical formulas, using either a series operating according to the powers of time, or a trigonometric series. However, this does not teach us anything about the nature of things, since it is a well known fact that any analytical function is expandable, within a finite domain, into a Taylor series or into a Fourier series. Furthermore, these methods are totally useless for preliminary calculations.

As the variations are highly complex in appearance, we naturally decide to look for simple laws, if there are any, not in the effect observed, but in the cause of the phenomena, i.e. in the distribution of the magnetic mass inside the earth. This distribution is expressed by arbitrariness shown in the expansion of the function of the earth's magnetic forces which precisely are the integrals defined in certain functions of the mass coordinates. It is clear that any theory must commence by a study of these arbitrariness.

This problem, seemingly very simple, in reality encounters serious problems due to the nature of the question itself, but also to the imperfection of the old observations. If the illustrious author himself of the Allgemeine Theorie des Erdmagnetismus despaired in being able to succeed with

*Numbers in the margin indicate pagination in the original text.

**Stockholm 1896

with the materials bequeathed to us by our predecessors, undoubtedly, it was a sort of scientific gourmandism, not too commonly found among scholars. Bessel is even more skeptical about this in his Populäre Vorlesungen über wissenschaftliche Gegenstände (P.339); 372). All contemporary authors have respected the authority of these scholars, and for good reason. Spoiled by the ever increasing precision of modern observations, we are too inclined to look down on traditional observations made with imperfect instruments, by unskilled observers, yet when they are combined in the appropriate manner, they are often called upon to render their good services.

The following lines intend to show that this task is possible, by presenting a simple method by which the objective can be reached without too much difficulty, and by presenting the already very satisfactory results that I obtained in the first application of this method.

As we were confronted only with unknowns in the subject in question, we had to begin by looking for the empirical laws of the phenomena by inductive reasoning. This is the first aspect that we will deal with.

From this, by elementary reasoning, we will deduce the magnetic field which produces each instant change in the earth's magnetization, then we will trace back to the physical causes which we think are responsible for the results obtained. Once we have found this principle, we will be able to express the differential equation that the solution of the question depends upon and we will see the main features revealed by experience follow from this at once.

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1.

The function of the forces of a sphere of attracting material, of radius R , for an outside point of distance r from the center of the sphere, may be expanded in a decreasing series with respect to the powers of r and therefore having this expression:

$$V = \frac{1}{r} \cdot Y^{(0)} + \frac{R}{r^2} \cdot Y^{(1)} + \frac{R^2}{r^3} \cdot Y^{(2)} + \text{etc.}$$

where the Laplace functions are represented. We know that $Y^{(i)}$ is a rational and whole function of μ , $\sqrt{1-\mu^2} \cdot \sin \varpi$, and $\sqrt{1-\mu^2} \cdot \cos \varpi$. where $\mu = \cos \theta$, which the vector radius r forms with the x, ϖ . axis. The is the angle which the plane passing through the radius and through this axis forms with the x and y plane.

Each of the $Y^{(i)}$ functions contains $2i+1$ arbitrary constants, each term is made up of $2i+1$ terms. The part of $Y^{(i)}$ depending on the $n\varpi$ angle will be

$$Y_n^{(i)} = (1 - \mu^2)^{\frac{n}{2}} \cdot \frac{d^n X_i}{d\mu^n} \cdot (A_n^{(i)} \sin n\varpi + B_n^{(i)} \cos n\varpi),$$

by letting X_i stand for the Legendre* polynomials. If we do $n = 0$, $n = 1$, $n = 2$, ... $n = i$, in this function successively, the sum of all the resulting functions will be the general expression of $Y^{(i)}$. The $A_n^{(i)}$ and $B_n^{(i)}$ are the numerical coefficients which are dependent upon the mass distribution and which must be derived in the experiment.

We are proposing an empirical study of the secular inequalities of arbitraries $A_n^{(i)}$ and $B_n^{(i)}$, based on accessible data.

Following the usual procedure, we will divide the

*We have retained the notations of la Mécanique Céleste [Celestial Mechanics] (book III, chapter II); if we wanted to use Gauss' notations, we would set $A_n^{(i)} = h^{i \cdot n}$, $B_n^{(i)} = g^{i \cdot n}$.

the attracting force S , for any given location on the earth's surface, into three rectangular axes, by adopting the plane of the horizon for the xy plane and the vertical plane for the z axis. Once this is established, the force components will be the partial derivatives of V with respect to these axes:

$$X = \sqrt{1-\mu^2} \cdot \frac{dV}{d\mu}, \quad Y = -\frac{1}{\sqrt{1-\mu^2}} \cdot \frac{dV}{d\varpi}, \quad Z = -\frac{dV}{dr}.$$

The components of the attraction are linear functions of the arbitraries $A_n^{(i)}$ and $B_n^{(i)}$, which may be determined if a sufficient number of data are used, for any time where the force is known.

The criteria were different for past centuries. Earlier observations did not give us the magnitude of the magnetic force, but only its direction. It is therefore necessary to establish equations dependent upon the declination and the dip of the magnetic needle and to select an expression from which the unknown values may be deduced.

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The most direct method of obtaining the differences from the constants $A_n^{(i)}$ and $B_n^{(i)}$ with respect to time would be to calculate them using the formulas which give the differences of the declination and the dip as a function of the components of force. The variations of the declination $d\delta$ and dip di , are linear functions of dX, dY and dZ , and therefore of $dA_n^{(i)}$ and $dB_n^{(i)}$, by virtue of the known formulas:*

$$\begin{aligned} R d\delta &= -\sin \delta \cdot dX + \cos \delta \cdot dY, \\ R di &= -\cos i \sin i \cos \delta \cdot dX - \cos i \sin i \sin \delta \cdot dY \\ &\quad + \cos^2 i \cdot dZ, \end{aligned}$$

* We obtain these formulas and the following differential formulas, by differentiating the system: $X = S \cos i \cos \delta$, $Y = S \cos i \sin \delta$, $Z = S \sin i$, or the system which we deduced by solving

δ and i ($R = S \cos i$) with respect to R .

where R stands for the horizontal component of the terrestrial magnetic force. These are the formulas used by Erman and Petersen to reduce the observations to a common date in order to revise the Gaussian calculations they performed and which we will return to later in this report.

Unfortunately, this method becomes unfeasible when a large number of terms are taken into consideration, because of the considerable amount of work required to solve the conditions equations. To clarify this, let us assume that we have to determine coefficients $A_n^{(i)}$ and $B_n^{(i)}$ up to and including the terms of the i^{th} order; we would then have to solve a system of equations with $i(i+2)$ unknowns. Moreover, the resolution of a large number of equations with 25, 35, 48...unknowns, using the least squares method, is certainly the type of task to discourage the most enterprising calculator, especially if he must repeat the process 20 times.

We might consider expanding $d\delta$ and di directly as a function of $dA_n^{(i)}$ and $dB_n^{(i)}$, and I admit that this is actually the method that I tried first in order to obtain the variations of $A_n^{(i)}$ and $B_n^{(i)}$. We show that the products $S \frac{dS}{dt}$, $S^2 \cos^2 i \frac{d\delta}{dt}$, $S^3 \cos i \frac{di}{dt}$, may be expanded from the spherical functions. Indeed, these products are made up of a series of terms having the expression $X_i \frac{dX_j}{dt}$ or $X_i X_j \frac{dX_k}{dt}$, X representing any one of the quantities X, Y, Z : these will be the expansion products according to the $Y_n^{(i)}$ functions, and these products themselves will be expanded into a series of $Y_n^{(i)}$. The coefficients of these expansions will be made up of terms from one of the expressions $A_n^{(i)} \frac{dA_m^{(i)}}{dt}$, $A_n^{(i)} A_m^{(j)} \frac{dA_p^{(k)}}{dt}$. The coefficients of these expansions may be expressed by set integrals, and since we therefore have enough data to solve

the problem, their values will be calculated independently of each other by simple quadratures. Moreover, as we are assuming that these polynomials are known, in order to deduce the values of $dA_n^{(i)}$ and $dB_n^{(i)}$, we will have to solve a system of $2i(2i+2)$ or $3i(3i+2)$ equations for $i(i+2)$ unknowns, and we will see the same difficulty appear again in a new form.

It was therefore necessary to use a simpler and more direct method. The method which we use consistently is a method of approximation, which consists of calculating the variations of the arbitraries $dA_n^{(i)}$ and $dB_n^{(i)}$, according to ordinary formulas which give the X, Y and Z components as a function of $A_n^{(i)}$ and $B_n^{(i)}$. This is achieved by adopting the values resulting from the first terms of the formulas below as the approximative values of dY /6
 dZ :

$$\begin{aligned} dY &= R \cos \delta \cdot d\delta + \sin \delta \, dR, \\ dZ &= \frac{R}{\cos^2 i} \cdot di + \tan i \cdot dR, \end{aligned}$$

the first terms are much larger than the second ones, if δ and i are small. We therefore disregard the variations of intensity which were not revealed by earlier observations.

A second approximation, in which R is variable (we may use the value resulting from the first calculation), will then give a more approximate value of $A_n^{(i)}$ and $B_n^{(i)}$, and by continuing in this way we may extend the approximation as far as we want to.

This method does not perhaps always offer the most reliable means of obtaining a calculation of the variations of the arbitraries, particularly with respect to

the dip. However, from the standpoint of application, it offers great simplicity.

2.

In the first approximation, we limited ourselves to the terms of the first two orders and therefore set:

$$V = \frac{R}{r^2} \cdot Y^{(1)} + \frac{R^2}{r^5} \cdot Y^{(2)},$$

by noting that for a magnetic body $Y^{(0)} = 0$.

The complete formulas used to determine the unknowns will be given later in the text (no. 6). Let us simply state here that we have followed the Gaussian method to perform our calculations, by evaluating the $\sin \mu$ and $\cos \mu$ coefficients in the expressions representing the forces by means of mechanical quadratures, after which the unknowns were divided into several groups in order to make it much more convenient to determine them.

The calculations were extended successively to earlier and earlier periods in history, starting with the year 1829: for the declination, from 1829 to 1787, from 1787 to 1700, then from 1700 to 1600; for the dip, from 1829 to 1787 and from 1787 to 1700.

Having obtained, with the accessible data, the values of the declination and the dip at the intersecting points of nine equidistant parallels, between the northern and southern latitude of 60° *, with 12 meridians in the longitudes 0° , 30° , 60° East of Greenwich, the differences $R \cos \delta \cdot d\delta$ or $\frac{R}{\cos^2 \delta} di$ were calculated. Having no knowledge at all of the intensity variations, we were forced to

*For 1600, and for the dip in 1787, observations of the parallel 60° south were lacking; for the dip of 1700 the parallels of 60° north and south were both lacking.

in this first approximation, the 1829 values for R , which inevitably would introduce errors that the subsequent approximations had to correct. These differences were then assimilated with dY and dZ , and substituted in the formulas of no. 6, reduced to their first terms. /7

After calculating the values of coefficients M and N using the method just described, I then found the variations $dA_n^{(i)}$ and $dB_n^{(i)}$ for the selected time intervals, which when successively added to the initial values for 1829, gave the values for $A_n^{(i)}$ and $B_n^{(i)}$ for the selected dates.

One important condition required for the validity of our method of approximation is that we can disregard the squares and the products of variations $dA_n^{(i)}$ and $dB_n^{(i)}$. This condition is far from being fulfilled for these long intervals of hundreds of years. We see to what extent the causes combine to distort the results, and we should expect no more than a very rough approximation of this first test.

3.

Our numerical calculations are based mainly on the magnetic charts accompanying Hansteen's work, Untersuchungen über den Magnetismus der Erde, published in Christiania in 1819. The very incomplete charts of this author for 1600 and 1700 were completed, in the Pacific, with information at our disposal.

For the 1600 chart, we took advantage of Tasman's observations, made in 1642 and 1643, near New Holland.

This included W. van Schouten's observation southeast of the Marquesas, reported by Hansteen. Another was Candish's observation, cited by Kircher in the appendix to the 2nd edition of his work Magnes, sive de Arte magnetica (Col. Agripp. 1643). Finally, we used another observation in the 3rd table of oceanic observations. Along the northern coast of America, my attention was drawn to a compass-card on a Wytfliet's chart, in Fascimileatlas by Nordenskiöld and which I identified after noticing an error in a sign.

For 1700, we have observations by Wooden Rogers during a trip from California to the Philippine Islands in 1710, and a few observations east of New Holland.

The chart of Wilcke's equal inclination lines is found in volume XXIX in Mémoires de l'Académie de Stockholm, 1768. The isoclinical lines of the North Atlantic are plotted here in full outlines according to Ekeberg and Lacaille's observations and in broken lines according to P. Fevillée's observations in 1710-1711. I followed these. In the Indian Ocean, Wilcke had plotted the isoclinical lines according to Ekeberg's observations in 1766. However, in these regions, I preferred earlier observations made by Cunningham in 1700. Finally, in the Pacific Ocean, a very precarious interpolation gave me very inadequate values of the inclination. I have to admit that this part of the curves is based more on conjectures than on reliable facts.

4.

I believe I have given a full account of the numbers found, as provided directly by the first calculation, although they have no other value than to show an example of how one can pass from total ignorance to a more and more

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rigorous knowledge of the truth:

The numerical values of the 8 constants of the first two orders, obtained by the first calculation, are shown below (the second lines corresponding to 1700 and 1787 refer to inclination observations):

Years	$B_0^{(1)}$	$A_1^{(1)}$	$B_1^{(1)}$
1600	—	—0.018058	+ 0.023456
1700	—	—0.032392	+ 0.032618
'	+ 0.288411	—0.049858	+ 0.080636
1784	+ 0.341369	—0.051882	+ 0.032882
1787	—	—0.049249	+ 0.028367
'	+ 0.325972	—0.042256	+ 0.051105
1829	+ 0.320074	—0.060109	+ 0.023353
1830	+ 0.323477	—0.062456	+ 0.031106
1858	+ 0.324627	—0.057268	+ 0.023641
1880	+ 0.333923	—0.061920	+ 0.027636
1885	+ 0.315720	—0.060258	+ 0.024814

Years	$B_0^{(2)}$	$A_1^{(2)}$	$B_1^{(2)}$	$A_2^{(2)}$	$B_2^{(2)}$
1600	—	—0.028352	—0.042494	+ 0.026955	+ 0.016715
1700	—	—0.012273	—0.036263	+ 0.005250	+ 0.014823
'	+ 0.023114	+ 0.006626	—0.006325	—0.000898	+ 0.022994
1784	+ 0.004694	—0.018639	—0.030099	—0.013829	+ 0.006553
1787	—	—0.015681	—0.037125	—0.005684	+ 0.007518
'	—0.014242	—0.011230	—0.031855	—0.011063	+ 0.008777
1829	+ 0.001210	+ 0.000720	—0.044537	—0.012637	+ 0.001249
1830	—0.007708	—0.002107	—0.050635	—0.013631	+ 0.000172
1858	—0.007190	—0.002226	—0.050827	—0.015093	—0.001337
1880	—0.005507	—0.003805	—0.051930	—0.012433	—0.004284
1885	+ 0.007906	+ 0.012999	—0.049798	—0.012604	—0.005667

We have inserted in this table the results of earlier calculations, performed before ours, and in particular by Gauss for 1830, by Erman and Petersen for 1829, by Mr. von Quintus Icilius, for 1880, and finally by Neumayer and Petersen, for 1885. We have included Erman's and Petersen's values for 1784 and 1858, deduced from the parabolic

formulas they gave in their report on page 11.

Here are the numbers which will serve as the basis for our subsequent calculations.

For those who are interested in referring to the original sources, the relevant bibliographical information is provided below:

- Gauss, Allgemeine Theorie des Erdmagnetismus (Gauss Werke, t. V; Resultate aus den Beobachtungen des magnetischen Vereins; 1838).
- Gauss & Weber, Atlas des Erdmagnetismus nach den Elementen der Theorie entworfen, Leipzig 1840.
- Goldschmidt, Vergleichung magnetischer Beobachtungen nach den Elementen der Theorie, (Resultate aus den Beobachtungen des magnetischen Vereins; 1840). /9
- H. Petersen, Vergleichung der gaussischen Theorie des Erdmagnetismus mit A. Ermans magnetischen Beobachtungen, (Astronomische Nachrichten, no.s 450, 452, 454).
- Erman & Petersen, Die Grundlagen der gaussischen Theorie und die Erscheinungen des Erdmagnetismus im Jahre 1829, Berlin 1874.
- G.Von Quintus Icilius, Der magnetische Zustand der Erde, (Aus dem Archiv des Deutschen Seewarte, 4th year, 1881, n. 2).
- Neumayer, Atlas des Erdmagnetismus, in the 4th section of the new edition of the Berghaus Atlas on physics, Gotha 1891.
- Ad. Schmidt, Mitteilungen über eine neue Berechnung des Erdmagnetischen Potentials. (Abhandlungen der königl. bayrischen Akademie der Wissenschaften, Vol. XIX, 1st part, Munich 1895.) (For translation see NASA-TM-77329).

5.

These values contain all sorts of errors due to the imprecision of the charts and to the influence of the terms omitted in the expansion of the $A_n^{(i)}$ and $B_n^{(i)}$ values according to the powers of time. In order to reduce the detrimental influence of these causes of errors as far as possible, it is necessary to associate the numbers found in a formula expressing the law of variation for constants $A_n^{(i)}$ and $B_n^{(i)}$.

We know nothing *a priori* about the form of quantities $A_n^{(i)}$ and $B_n^{(i)}$ taken into consideration as a function of time. It seems natural at first to expand them into a series according to the powers of time, and to stop the expansion at a power value that is not too high. However, a moment's reflection will tell us that these quantities cannot contain secular terms per se in which time is a factor. Actually, this would imply that the quantity $\sqrt{A_n^{(0)^2} + B_n^{(0)^2}}$, and consequently the corresponding magnetic moment could become as large as we want it to become, which is impossible.

It is also not very probable that they will get close to a limit, as we may assume that the earth's magnetization dates back far enough for a steady state to be reached in the variations.

The only probable hypothesis is that they do not really contain any periodic terms. One of the simplest hypotheses that we can make in this regard is that each pair of transcendental quantities $A_n^{(i)}$ and $B_n^{(i)}$ has cycles which last the same period of time. It remains to be seen that this hypothesis is verified by experience.

Let us replace the quantities $A_n^{(i)}$ and $B_n^{(i)}$, $i(i+2)$

by new transcendental quantities $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ related to the first ones by equations having the expression:

$$B_n^{(i)} - A_n^{(i)} \sqrt{-1} = \alpha_n^{(i)} \cdot E^{\beta_n^{(i)} \sqrt{-1}},$$

where E represents the base of the Napierian logarithms. $\alpha_n^{(i)}$ will be the module or the magnitude of the magnetic moment $\sqrt{A_n^{(i)2} + B_n^{(i)2}}$, angle $\beta_n^{(i)}$ will distinguish its direction in space.

Once this is established, the part of $Y^{(i)}$ depend-
end upon angle $n\varpi$, will be expressed:

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$$(1 - \mu^2)^{\frac{n}{2}} \frac{d^n X_i}{d\mu^n} \cdot \alpha_n^{(i)} \cos n(\varpi + \beta_n^{(i)}).$$

We will look for the law of variation for the new constants $\alpha_n^{(i)}$ and $\beta_n^{(i)}$.

The numerical values of the constants $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ derived from the values of $A_n^{(i)}$ and $B_n^{(i)}$ found in the preceding number are shown below (the $\beta_n^{(i)}$ may always be selected in such a manner that the values of $\alpha_n^{(i)}$ will be essentially positive, if $n > 0$):

The values of the coefficients $\alpha_o^{(1)}$ and $\alpha_l^{(2)}$ are identical to those of $B_o^{(1)}$ and $B_o^{(2)}$ given earlier.

These numbers show more clearly the law of variation of the arbitraries $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ and we already guess that this law is very simple. Actually, the variations seem to be proportional to time. It is the function $Y_2^{(2)}$ in particular which gives us some evidence: coefficient $^{(2)}_2$

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*We may assume that this angle is equal to 133.67, if we give the corresponding coefficient $\alpha_l^{(2)}$ a negative sign. The value of $\beta_o - \beta_o$ given in the next column (p.14) refers to the value of $\beta_l^{(2)}$ minus 180°.

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remains fairly constant, while angle $\beta_2^{(2)}$ has shifted nearly a quarter of a circle over the past three centuries and this means that if it keeps up at this rate it will take a total of 1200 years to move around in a full circle.

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Years	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\beta_1^{(1)}$	$\beta_0 - \beta_0$
1600	—	+ 0.029602	+ 37°.60	— 7°.18
1700	—	+ 0.045970	+ 44°.80	— 1°.34
,	+ 0.288411	+ 0.094805	+ 31°.73	+ 11°.73
1784	+ 0.341369	+ 0.061424	+ 57°.53	— 3°.20
1787	—	+ 0.056836	+ 60°.05	— 5°.23
,	+ 0.325972	+ 0.066313	+ 39°.58	+ 15°.24
1829	+ 0.320074	+ 0.066461	+ 64°.75	— 4°.45
1830	+ 0.323477	+ 0.069773	+ 63°.52	— 3°.09
1858	+ 0.324627	+ 0.061956	+ 67°.57	— 3°.48
1880	+ 0.333923	+ 0.067808	+ 65°.95	+ 1°.01
1885	+ 0.315720	+ 0.065169	+ 67°.62	— 0°.01

Years	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$	$\beta_1^{(2)}$	$\beta_0 - \beta_0$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\beta_0 - \beta_0$
1600	—	+ 0.051084	+ 146°.23	— 17°.70	+ 0.031724	— 29°.10	— 1°.76
1700	—	+ 0.038284	+ 161°.30	— 13°.15	+ 0.014789	— 10°.40	+ 10°.11
,	+ 0.023114	+ 0.009160	+ 313°.67	+ 14°.48	+ 0.023011	+ 1°.12	— 1°.41
1784	+ 0.004694	+ 0.035404	+ 148°.23	+ 16°.37	+ 0.015303	+ 32°.33	— 6°.93
1787	—	+ 0.040301	+ 157°.10	+ 8°.08	+ 0.009425	+ 18°.55	+ 7°.76
,	— 0.014242	+ 0.033777	+ 160°.55	+ 4°.60	+ 0.014122	+ 25°.79	+ 0°.52
1829	+ 0.001210	+ 0.044543	+ 180°.94	— 7°.53	+ 0.012699	+ 42°.18	— 3°.02
1830	— 0.007708	+ 0.050677	+ 177°.62	— 4°.02	+ 0.013632	+ 44°.64	— 5°.18
1858	— 0.007190	+ 0.050876	+ 177°.50	+ 1°.58	+ 0.015152	+ 47°.54	+ 0°.48
1880	— 0.005507	+ 0.052068	+ 175°.82	+ 7°.56	+ 0.013151	+ 54°.51	+ 0°.25
1885	+ 0.007906	+ 0.051468	+ 194°.65	— 10°.26	+ 0.013819	+ 57°.11	— 0°.83

This leads us to represent the variations of the constants using linear formulas of time. We have found that if we give each determination the same value:

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$$\begin{aligned} \alpha_0^{(1)} &= +0.322192 + 50.10 t, & \beta_1^{(1)} &= 50.30 + 0.1305 t, \\ \alpha_1^{(1)} &= +0.064528 + 49.46 t, & \beta_1^{(2)} &= 173.40 + 0.1957 t, \\ \alpha_0^{(2)} &= -0.000389 - 68.21 t, & \beta_2^{(2)} &= 39.15 + 0.3058 t, \\ \alpha_1^{(2)} &= +0.045311 + 123.36 t, \\ \alpha_2^{(2)} &= +0.013577 - 57.36 t; \end{aligned}$$

Time is counted as of 1829, and the variable terms of the modules are expressed in units of the sixth decimal order.

Coefficients $\alpha_n^{(i)}$ vary very slowly with time, and it is permissible to assume, within the limits of observation errors, that the time factor is strictly zero. The very large discrepancy between the numbers found do not enable us to determine whether the $\alpha_n^{(i)}$ modules still contain periodical terms. However, for theoretical reasons which we will give later (no. 15), it is probable that the periodical inequalities which may exist in $\alpha_n^{(i)}$ are of the same order of magnitude as the periodical inequalities of angles $\beta_n^{(i)}$. As the observations have still not revealed any such thing, we have decided to set:

$$\alpha_n^{(i)} = \text{const.},$$

and therefore:

$$\beta_n^{(i)} = \gamma_n^{(i)} + m_n^{(i)} t,$$

$\gamma_n^{(i)}$ and $m_n^{(i)}$ being two new constants.

By determining the $\alpha_n^{(i)}$ according to the six most accurate modern determinations, and by excluding previous observations which mainly produce the term in t , we have finally set:*

*A comparison of these values with the small table below shows that the declination chart for 1600, and that of the inclination for 1700 are highly inaccurate, as we could naturally assume.

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$$\begin{aligned} \alpha_0^{(1)} &= +0.326533, & - \\ \alpha_1^{(1)} &= +0.065432, & \beta_1^{(1)} = 60.30 + 0.1305 t, \\ \alpha_0^{(2)} &= -0.001099, & - \\ \alpha_1^{(2)} &= +0.047506, & \beta_1^{(2)} = 173.40 + 0.1957 t, \\ \alpha_2^{(2)} &= +0.013959, & \beta_2^{(2)} = 39.15 + 0.3058 t. \end{aligned}$$

These will serve as the base values for our second approximation, in the calculation of the $A_n^{(i)}$'s & $B_n^{(i)}$'s as of the year 1538 to and including 1800.

6.

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Once we have found our first approximation, it will be extended as far as possible. The same formulas are applicable to the case already processed, by letting the characteristic d no longer be the difference with respect to time, but the correction which must be made of the approximations of the unknown values.

To these rectifying calculations, we have extended the series of the potential up to and including the terms of the fourth order, which are indispensable for a correct representation of the facts.

These complete formulas which were used to calculate the $DA_n^{(i)}$'s and the $dB_n^{(i)}$'s according to the dX s, dY s and dZ s will serve at this point.

If we set:

$$P_n^{(i)} = (1 - \mu^2)^{\frac{n}{2}} \frac{d^n X_i}{d\mu^n},$$

the general term of the function for the magnetic forces will be (no. 1):

$$\frac{R^i}{r^{i+1}} P_n^{(i)} (A_n^{(i)} \sin n\varpi + B_n^{(i)} \cos n\varpi).$$

In the expressions of the components, the terms corresponding to any given point on the terrestrial surface will be:

$$\begin{aligned} & \sqrt{1-\mu^2} \frac{dP_n^{(i)}}{d\mu} (A_n^{(i)} \sin n\varpi + B_n^{(i)} \cos n\varpi); \\ & - \frac{n}{\sqrt{1-\mu^2}} P_n^{(i)} (A_n^{(i)} \cos n\varpi - B_n^{(i)} \sin n\varpi); \\ & (i+1) P_n^{(i)} (A_n^{(i)} \sin n\varpi + B_n^{(i)} \cos n\varpi). \end{aligned}$$

If we call L_n , L'_n , M_n , M'_n , N'_n the coefficients of $\cos n\varpi$ and $\sin n\varpi$ in the expansion of X , Y and Z according to the multiples of ϖ , the equations which determine L_n , L'_n , M_n , etc., will be:

$$\begin{aligned} L_n &= \sum \sqrt{1-\mu^2} \frac{dP_n^{(i)}}{d\mu} B_n^{(i)}; & M_n &= - \sum \frac{n}{\sqrt{1-\mu^2}} P_n^{(i)} A_n^{(i)}; & N_n &= \sum (i+1) P_n^{(i)} B_n^{(i)}; \\ L'_n &= \sum \sqrt{1-\mu^2} \frac{dP_n^{(i)}}{d\mu} A_n^{(i)}; & M'_n &= \sum \frac{n}{\sqrt{1-\mu^2}} P_n^{(i)} B_n^{(i)}; & N'_n &= \sum (i+1) P_n^{(i)} A_n^{(i)}. \end{aligned}$$

By focussing on the terms up to and including the fourth order, we will find in particular the following formulas:

$$L_0 = \sqrt{1-\mu^2} \cdot B_0^{(1)} + 2\mu\sqrt{1-\mu^2} \cdot B_0^{(2)} + (3\mu^2 - \frac{1}{3})\sqrt{1-\mu^2} \cdot B_0^{(3)} + (4\mu^3 - \frac{1}{3}\mu)\sqrt{1-\mu^2} \cdot B_0^{(4)},$$

$$N_0 = 2\mu \cdot B_0^{(1)} + 3(\mu^2 - \frac{1}{3}) \cdot B_0^{(2)} + 4(\mu^3 - \frac{1}{3}\mu) \cdot B_0^{(3)} + 5(\mu^4 - \frac{1}{3}\mu^2 + \frac{1}{35}) \cdot B_0^{(4)};$$

$$L'_1 = -\mu \cdot A_1^{(1)} - (2\mu^2 - 1) \cdot A_1^{(2)} - (3\mu^3 - \frac{1}{3}\mu) \cdot A_1^{(3)} - (4\mu^4 - \frac{2}{3}\mu^2 + \frac{1}{3}) \cdot A_1^{(4)},$$

$$M'_1 = -A_1^{(1)} - \mu \cdot A_1^{(2)} - (\mu^2 - \frac{1}{3}) \cdot A_1^{(3)} - (\mu^3 - \frac{1}{3}\mu) \cdot A_1^{(4)},$$

$$N'_1 = 2\sqrt{1-\mu^2} \cdot A_1^{(1)} + 3\mu\sqrt{1-\mu^2} \cdot A_1^{(2)} + 4(\mu^2 - \frac{1}{3})\sqrt{1-\mu^2} \cdot A_1^{(3)} + 5(\mu^3 - \frac{1}{3}\mu)\sqrt{1-\mu^2} \cdot A_1^{(4)};$$

$$L_1 = -\mu \cdot B_1^{(1)} - (2\mu^2 - 1) \cdot B_1^{(2)} - (3\mu^3 - \frac{1}{3}\mu) \cdot B_1^{(3)} - (4\mu^4 - \frac{2}{3}\mu^2 + \frac{1}{3}) \cdot B_1^{(4)},$$

$$M'_1 = B_1^{(1)} + \mu \cdot B_1^{(2)} + (\mu^2 - \frac{1}{3}) \cdot B_1^{(3)} + (\mu^3 - \frac{1}{3}\mu) \cdot B_1^{(4)},$$

$$N_1 = 2\sqrt{1-\mu^2} \cdot B_1^{(1)} + 3\mu\sqrt{1-\mu^2} \cdot B_1^{(2)} + 4(\mu^2 - \frac{1}{3})\sqrt{1-\mu^2} \cdot B_1^{(3)} + 5(\mu^3 - \frac{1}{3}\mu)\sqrt{1-\mu^2} \cdot B_1^{(4)};$$

$$L'_2 = -2\mu\sqrt{1-\mu^2} \cdot A_2^{(2)} - (3\mu^2 - 1)\sqrt{1-\mu^2} \cdot A_2^{(3)} - (4\mu^3 - \frac{1}{3}\mu)\sqrt{1-\mu^2} \cdot A_2^{(4)},$$

$$M'_2 = -2\sqrt{1-\mu^2} \cdot A_2^{(2)} - 2\mu\sqrt{1-\mu^2} \cdot A_2^{(3)} - 2(\mu^2 - \frac{1}{3})\sqrt{1-\mu^2} \cdot A_2^{(4)},$$

$$N'_2 = 3(1-\mu^2) \cdot A_2^{(2)} + 4\mu(1-\mu^2) \cdot A_2^{(3)} + 5(\mu^2 - \frac{1}{3})(1-\mu^2) \cdot A_2^{(4)};$$

$$L_2 = -2\mu\sqrt{1-\mu^2} \cdot B_2^{(2)} - (3\mu^2 - 1)\sqrt{1-\mu^2} \cdot B_2^{(3)} - (4\mu^3 - \frac{1}{3}\mu)\sqrt{1-\mu^2} \cdot B_2^{(4)},$$

$$M'_2 = 2\sqrt{1-\mu^2} \cdot B_2^{(2)} + 2\mu\sqrt{1-\mu^2} \cdot B_2^{(3)} + 2(\mu^2 - \frac{1}{3})\sqrt{1-\mu^2} \cdot B_2^{(4)},$$

$$N_2 = 3(1-\mu^2) \cdot B_2^{(2)} + 4\mu(1-\mu^2) \cdot B_2^{(3)} + 5(\mu^2 - \frac{1}{3})(1-\mu^2) \cdot B_2^{(4)};$$

$$L'_3 = -3\mu(1-\mu^2) \cdot A_3^{(3)} - (4\mu^2 - 1)(1-\mu^2) \cdot A_3^{(4)},$$

$$M'_3 = -3(1-\mu^2) \cdot A_3^{(3)} - 3\mu(1-\mu^2) \cdot A_3^{(4)},$$

$$N'_3 = 4(\sqrt{1-\mu^2})^3 \cdot A_3^{(3)} + 5\mu(\sqrt{1-\mu^2})^3 \cdot A_3^{(4)};$$

$$L_3 = -3\mu(1-\mu^2) \cdot B_3^{(3)} - (4\mu^2 - 1)(1-\mu^2) \cdot B_3^{(4)},$$

$$M'_3 = 3(1-\mu^2) \cdot B_3^{(3)} + 3\mu(1-\mu^2) \cdot B_3^{(4)},$$

$$N_3 = 4(\sqrt{1-\mu^2})^3 \cdot B_3^{(3)} + 5\mu(\sqrt{1-\mu^2})^3 \cdot B_3^{(4)};$$

$$L'_4 = -4\mu(\sqrt{1-\mu^2})^3 \cdot A_4^{(4)},$$

$$M'_4 = -4(\sqrt{1-\mu^2})^3 \cdot A_4^{(4)},$$

$$N'_4 = 5(1-\mu^2)^2 \cdot A_4^{(4)};$$

$$L_4 = -4\mu(\sqrt{1-\mu^2})^3 \cdot B_4^{(4)},$$

$$M'_4 = 4(\sqrt{1-\mu^2})^3 \cdot B_4^{(4)},$$

$$N_4 = 5(1-\mu^2)^2 \cdot B_4^{(4)}.$$

My first test made me aware that the charts I used were much too inaccurate, the large gaps left behind by ancient observations were filled more by bold conjectures than by verified facts. Specifically, when I misinterpreted a compass-card on Wytfliet's chart, I gave the Pacific coast of northern America a western declination rather than an eastern declination and thus made a serious error which calculations quickly corrected.

Since then, I found several ancient observations which enabled me to construct new charts for the XVIth and XVIIth centuries, and specifically for the years 1538, 1572, 1600, 1642, 1676 and 1700. The charts for 1600 and 1700 encompassed a complete region of the earth, including all longitudes. For the other periods, the dates needed in the Pacific were lacking: they encompassed only the region of the earth between longitude 90° West and longitude 150° East.

We will not reproduce these charts containing too many errors, we will simply show later in the text the results of a comparison of the chart for 1600 with theory.

We will succinctly indicate the main observations which served to construct the new magnetic charts.

The first of our charts is based mainly on the observations made by D. Joao de Castro during his trip from Lisbon to Goa in 1538 and to the Red Sea in 1541 (Roteiro de Lisboa a Goa, published by Joao de Andrade Corvo in Lisbon in 1882; Primeiro Roteiro de Costa da India, ed. Diogo Köpke, Porto 1843; Roteiro ao mar Roxo no anno de 1541, Paris 1833). The second group of charts are based on the

observations of the famous Christopher Columbus during his third voyage. The charts are also based on various notes from Sebastian Cabot's voyages and on Magellan's first voyage around the world (Pigafetta, Navigazione in torno al globo, reprinted in Raccolta di domuneti etc. dalla R. commissione colombiana, Part V, Vol. III, or in Atti della Socita ligure di Storia patrie, Genova, t. XV, p. 31). Finally, our charts are based on a certain number of data on the cities of Europe given by Tanstetter, Pierre Apien, George Hartmann, Joachim Rheticus, Mercator, Bellarmatus, Orance Finé, Cosimo Bartoli, etc., and others who we could mention, but which would make our list too long if we gave them all here.

The major gaps left by these observations in vast regions can be filled only by a very precarious process. For this purpose, I used ancient mariner's charts. We know that the ancient navigators were accustomed to constructing their mariners charts according to compass indications without being concerned about variations in the magnetic needle. By comparing the ancient mariners charts with modern charts, we can deduce the declination of the magnet. Among these charts, we will mention those which existed in Harrisse's work (Discovery of the North America, Paris 1892): the maps of la Cosa, Cantino, Vesconte Maggiolo, Estevam Gomez, Canerio, and a planisphere kept in the library of the king of Italy in Turin, dating back to the very first years of the XVIth century. We may also consult fruitfully the Facsimileatlas by Nordenskiöld, passim. and especially the charts of Bernardus Sylvanus, Martin Behaim's globe, etc.

The following charts are based on somewhat more reliable observations. For 1572, we have the observations of Vincente Rodrigues, published by Andrade Corvo (*loc. cit.*), according to a manuscript at the royal library of

Madrid. We also borrowed from a list of declinations extracted from Art de trouver les ports [The Art Of Finding Ports] by Simon Stevinus. [I had under my eyes Edw. Wright's translation of the third edition of his Treatise Certain Errors in Navigation Detected and Corrected (London 1657)]. This Table was often reproduced by the major compilers of magnetic observations of the XVIIth century, Gilbert, Kircher and Riccioli. We will also join a few facts on Adrian Metius given to us by P. Kircher in his famous work (Magnes, sive de Arte magnetica; p. 429 edition of Rome 1641). A certain number of European observations taken here and there will complete the chart for this date.

In regard to the next chart, for 1600, we tried to make it as complete as possible, and were essentially successful in our attempt, thanks to the observations in the South Sea available in Dudley's sea Atlas, called Arcano del Mare, Florence 1646-1647, and particularly to those of the southern hemisphere made by van Schouten and Lemaire in 1616. In the Old World, we have borrowed from observations made by Gaspar Remao in 1598 and published by Andrade Corvo as an appendix to D. Joao de Castro's voyage. We also borrowed observations by Thomas Candish, Hall, Davis and Linschot, in P. Kircher's Magnes (appendix to the second edition, Col. Agripp. 1643), an a large number of observations by Middleton, Daunton, Hippon, Saris, Castleton, Marlowe, etc., in the 3rd volume by Purchas [His Pilgrims (London 1625), quoted from Hansteen, Untersuchungen über den Magnetismus der Erde, Christiana, 1819.].

For the other two charts of the XVIth century, we will find various observations reported in general works of the period, by Kircher, Riccioli, Fournier, Semmlerus.

For the first half of the XVIIth century, we will

cite observations made by Fournier and Guerard (G. Fournier, Hydrographie [Hydrography], Paris 1643). Guerardus Diepensis' observations are also cited in Riccioli (Geographia et Hydrographia reformata, Venetiis 1672), who also copied all of his predecessors, Kircher, Dudleyaeus, Stenvinus, Janssønus, etc. We then offer two important observations by Olearius in the Caspian Sea (Delisle, Histoire de l'Académie, [History of the Academy] 1721.) Tasman's observations date back to the same period. For the North Atlantic, consult Richard Norwood, The Sea-mans Practice, London 1659. In Europe, we borrowed very numerous observations gathered by the diligent P. Athanase Kircher, and published by him in his work already cited so often, as well as observations in the Mediterranean basin reported by the same author.

The 1676 chart is partially extracted from the same sources as the previous one, and especially from Christophorus Semmierus' book, Methodus inveniendae longitudinis maritimae, Halae Magdeburgicae 1723. Let us also mention, for the end of the XVIIth century, Halley's Table in the Philosophical Transactions of London (volume XIII, 1683); Leydeker's observations in 1675 (Hansteen, Untersuchungen etc.); and a host of French Jesuit observations made in France, in China and during the trip to China (Anciennes Mémoires de l'Académie, t. VII, passim.) [Ancient Reports of the Academy].

Our chart for the year 1700 is, in all of its main aspects, a well-known reproduction of the Table by Moutaine and Dodson in the Transactions philosophiques de 1757 [Philosophical Transactions of 1757], p. 329. In the Pacific Ocean, the curves were plotted according to the voyage of Captain Wood Rogers 1710 (Transactions philosophiques, 1721, p. 173), and in the northern part following the voyage of the ship Navire St Antoine in 1707,

based on an interesting description of this voyage preserved in a manuscript at the Royal Library of Stockholm and which I became aware of through Mr. Dahlgren. The observations of Bering at Kamchatka made during his voyage in 1728, are extracted from handwritten charts of Bering, also preserved in the Royal Library. Inside Siberia, we have only Gmelin's observations in 1735, and a few notes by Euler, of a fairly uncertain date, plus van Verden's observations in the Caspian sea (*Histoire de l'Académie* 1718). For Europe, consult Musschenbroek (*Dissertatio de Magnete*, Viennae 1754), which also reports a host of nautical observations for the turn of the XVIIIth century. By including a certain number of observations in the Hudson Bay in Musschenbroek and a few other American observations in Ch.-A Schott (Secular variation of the magnetic declination in the United States; Report of the U.S. Coast and Geodetic Survey, 1888, App. n. 7, 7th edition, Washington 1890), we believe we have indicated the most important sources, offering our chart enough background to give it a broader extension, something which was not possible in Halley's time.

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Here are the main sources from which we have extracted our information. We could not invite the reader to follow us in all the often tedious research details which we went through to establish the location or date of the observations which ancient authors often described with vague details.

In conclusion, I would like to point out that in the charts which I had decided to sketch, I did not use any of the numerous and precious observations recently discovered by Mr. van Bemmelen, in the Dutch Record Office*, in fact I was unaware of his work when I set out to

*See his inaugural thesis presented before the Science Faculty at Leyde, 1893.

to construct my charts. Fortunately, there is no reason to be overly regretful about this unintentional omission, as I am convinced that the probable differences between his charts and mine are generally between 1° and 2° . This is the error range to be expected in ancient observations, and the errors which may be found in our conclusions are of little consequence.

Besides the six new charts which we had undertaken to sketch, we referred to Hansteen's well-known Atlas where equal declination lines will be found for the years 1710, 1720, 1730, 1744, 1756, 1770, 1780, 1787 and 1800, together with Wilcke's inclination chart for 1700-1760.

Finally, three new determinations of the potential for the XIXth century will complete the series of unknown values which will serve as a base for our studies. These determinations for the years 1820, 1840, and 1860, are based on the following data:

1. Hansteen's declination chart (K. Danske Videnskabs-Selskabs skrifter, 5th series, t. IV; 1855), and the inclination chart by the same author for 1827 (Annales de Poggendorff, t. XCVII; 1830), attached to Duperrey's total intensity chart, based on all intensity observations made from 1784 to 1828 (Bergahus' physikalischer Atlas, 2nd edition.)

2. Sabine's total intensity charts for 1837-1840, attached to the declination and inclination charts by the same author for 1840. The chart of isodynamic lines appeared first in the Report of the British Association, 1837. The declination chart is found in Johnston's Physical Atlas. All of these charts are reproduced in F. Walker, Terrestrial and cosmical magnetism, Cambridge

1866.

3. Finally, Evan's charts of the three elements of terrestrial magnetism for 1860 (Smith, Admiralty manual of the Deviations of the *, 3rd edition, London 1869).

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The calculations of the $A_n^{(i)}$ s and $B_n^{(i)}$ s based on these data were performed according to the same procedure used for the first approximation, by utilizing the complete formulas of no. 6, and by following the same calculating rules. These will not be repeated here.

However, the incomplete charts, where the Pacific was missing, required another method for calculating the coefficients L , M and N which could no longer be obtained using ordinary quadratures. We have obtained the values of the nine unknowns by solving the system of nine equations provided by the points of intersection of the nine meridians with each parallel.

These equations have constant coefficients, and it is possible to benefit from this situation to abbreviate the calculations. Let us call n_1, n_2, \dots, n_9 the numerical values of the second members of the condition equations for $\varpi = 0^\circ, 30^\circ, \dots, 150^\circ, 270^\circ, 300^\circ, 330^\circ$, used in the order just enumerated. We will immediately express the coefficients of these equations. By solving them, we obtain the following result where the coefficients are logarithms:

*Illegible.

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$$\begin{aligned}
 L_0 &= -0.3336 n_1 + 0.3960 n_2 - 0.3336 n_3 + 0.2711 n_4 - 0.0324 n_5 + 1.7942 n_6 + 1.7939 n_7 - 0.0326 n_8 + 0.2711 n_9 \\
 L_1 &= +0.5721 - 0.5721 + 0.5721 - 0.4690 + 0.2710 - 0.0327 - 1.8971 + 0.2711 - 0.4242 \\
 L_1' &= +0.3336 - 0.3335 + 0.3335 - 0.1357 + 0.0324 - 1.4612 - 1.8970 + 0.0325 - 0.2711 \\
 L_2 &= -0.1356 + 0.2710 - 0.2710 + 0.1356 - 0.0730 + 1.8346 - \infty - 1.8346 + 0.0732 \\
 L_2' &= -0.4690 + 0.5096 - 0.4242 + 0.3336 - 0.1679 + 1.5966 + 1.8970 - 0.2458 + 0.3544 \\
 L_3 &= -\infty - \infty - \infty - 1.4603 + 1.6990 - 1.4603 + 1.4606 - 1.6991 + 1.4606 \\
 L_3' &= +0.3335 - 0.3335 + 0.3335 - 0.2711 + 0.0325 - 1.4607 - 1.4618 + 0.0326 - 0.2710 \\
 L_4 &= +1.8970 - 1.7939 + 1.4606 - \infty - 1.0237 + 2.7846 - 1.0860 + 1.5962 - 1.8346 \\
 L_4' &= -1.8970 + 0.0324 - 0.0324 + 1.8970 - 1.5959 + 1.0241 - \infty - 1.0249 + 1.5960
 \end{aligned}$$

Similar equations determine the values of M_n , M_n' , N_n , N_n' .

Besides the four new charts, this method was used again for Hansteen's charts 1720, 1730 and 1756 and for Wilcke's chart for 1700-1760. The rest of the calculations were performed as in the preceding case.

The results of these calculations, in the form directly obtained, are included in the following tables (p. 18-19). These are fundamental data, directly deduced from the observations, which will serve as the base for all of our subsequent reasonings.

We will make the following remark in regard to these figures. Wilcke's inclination chart for 1700-1760 can hardly be used to determine the terms of the first two orders. This is because the insertion of higher order terms distorts the result. The numbers given below are those resulting from this assumption and where 3rd and 4th order terms are omitted.*

*We will cite, for the sake of curiosity, the numbers which we had obtained first, by using the complete formulas, for the independent functions of longitude:

Dates.	$B_0^{(1)}$	$B_0^{(2)}$	$B_0^{(3)}$	$B_0^{(4)}$
1700	+ 0.129781	+ 0.143782	- 0.321506	+ 0.153582
1760	+ 0.387062	+ 0.244060	+ 0.055931	+ 0.623814

These results, which are far from being accurate, show that we should be very cautious about giving expansions a broader extension than the observations can substantiate.

Next, we calculated the module $\alpha_n^{(i)}$ and the argument $\beta_n^{(i)}$ of each term, using the formulas of no. 5. The results of this work are shown on the next four pages.

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In order to examine the hypothesis offered above on the law of variation of the transcendental quantities $A_n^{(i)}$ and $B_n^{(i)}$, we have calculated, as before, the coefficient $\alpha_n^{(i)}$ and the argument $\beta_n^{(i)}$ of each term. The results of this work are shown on the next four pages.

These tables enable us to account for the variation of the unknowns with time. The linear variation of the arguments $\beta_n^{(i)}$ are obviously confirmed for the terms of the first two orders. In regard to higher order terms, the results are less obvious. However, as a general rule, we can affirm that the results do confirm the hypothesis used as the base for the second approximation, so that we may now set:

$$\alpha_n^{(i)} = \text{const.}; \quad \beta_n^{(i)} = \gamma_n^{(i)} + m_n^{(i)} t;$$

where t represents the number of years as of 1800.

If we refer to the meaning of the quantities $\alpha_n^{(i)}$ and $\beta_n^{(i)}$, the law of variation of the earth's magnetization can be expressed succinctly in the following words:

"The modules of each term of the magnetic force function remain constant, while their arguments increase proportionally with time."

The numerical values of the initial coefficients $\alpha_n^{(i)}$ and arguments $\beta_n^{(i)}$ and their yearly motions $m_n^{(i)}$ are shown

years	$B_0^{(1)}$	$A_1^{(1)}$	$B_1^{(1)}$	$B_0^{(2)}$	$A_1^{(2)}$	$B_1^{(2)}$
1538	—	—0.044672	+0.056277	—	—0.068463	—0.005050
1572	—	—0.062541	+0.056933	—	+0.000331	—0.045492
1600	—	—0.021374	+0.032864	—	—0.033729	—0.030416
1642	—	—0.093116	+0.065350	—	—0.031050	—0.028134
1676	—	—0.075413	+0.068606	—	—0.024094	—0.026200
1700	—	—0.028620	+0.032724	—	—0.022398	—0.033977
id.	+0.288251	—0.059880	+0.072543	—0.063404	—0.026486	+0.083313
1710	—	—0.031975	+0.032523	—	—0.014468	—0.027477
1720	—	—0.054521	+0.040967	—	—0.037366	—0.028640
1730	—	—0.045669	+0.033566	—	—0.042686	—0.025216
1744	—	—0.041497	+0.032746	—	—0.025255	—0.040958
1756	—	—0.036876	+0.024513	—	—0.094182	+0.005257
1760	+0.347964	+0.004396	+0.061003	+0.024643	—0.109768	—0.094125
1770	—	—0.047369	+0.031031	—	—0.014586	—0.037508
1780	+0.319216	—0.044853	+0.047254	—0.006100	—0.008785	—0.036419
1787	—	—0.050481	+0.031015	—	—0.015738	—0.038748
1800	—	—0.052754	+0.034423	—	—0.010200	—0.045947
1820	+0.333810	—0.062973	+0.026552	+0.005092	—0.000091	—0.030416
1829	+0.320074	—0.060109	+0.028355	+0.001210	+0.000720	—0.044537
1830	+0.323477	—0.062456	+0.031106	—0.007708	—0.002107	—0.050635
1840	+0.322334	—0.063180	+0.028641	—0.000354	—0.001905	—0.048071
1860	+0.325760	—0.059894	+0.029550	—0.001917	+0.001795	—0.047563
1880	+0.333923	—0.061920	+0.027625	—0.005507	—0.003805	—0.051930
1885	+0.315720	—0.060258	+0.024514	+0.007906	+0.012999	—0.049798

years	$A_2^{(2)}$	$B_2^{(2)}$	$B_0^{(3)}$	$A_1^{(3)}$	$B_1^{(3)}$	$A_2^{(3)}$	$B_2^{(3)}$
1538	+0.028063	—0.028945	—	—0.067942	—0.025203	—0.000341	—0.067016
1572	+0.035770	—0.011402	—	—0.029237	+0.003276	—0.010857	+0.000713
1600	+0.021534	+0.011750	—	—0.012933	+0.016891	+0.001655	—0.011415
1642	+0.040967	—0.018373	—	—0.114577	—0.032849	—0.008689	—0.008044
1676	+0.021315	—0.006077	—	—0.058067	—0.007136	—0.006700	—0.017533
1700	+0.004272	+0.017030	—	—0.003912	+0.007041	+0.001426	—0.021382
id.	—0.001686	+0.019182	—	—	—	—	—
1710	+0.001774	+0.016588	—	—0.008499	+0.000764	+0.000012	—0.009500
1720	+0.006400	+0.010162	—	—0.052379	+0.020753	+0.008006	—0.025445
1730	+0.001351	+0.012930	—	—0.047195	+0.048819	+0.000566	—0.024583
1744	—0.002059	+0.018038	—	—0.006576	+0.001077	+0.002456	—0.017374
1756	—0.004824	+0.014594	—	—0.024272	—0.008624	+0.016110	—0.057395
1760	—0.062791	—0.015726	—	—	—	—	—
1770	—0.004824	+0.008602	—	—0.011707	+0.027345	—0.006485	—0.026892
1780	—0.012609	+0.026787	—0.016383	—0.019010	+0.033642	—0.003230	—0.020201
1787	—0.006133	+0.007939	—	—0.016016	+0.024783	—0.006418	—0.027758
1800	—0.008586	+0.008619	—	—0.026726	+0.022926	—0.003822	—0.021547
1820	—0.014513	—0.001898	—0.001792	—0.002657	+0.027497	—0.005187	—0.015466
1829	—0.013637	+0.001249	—0.018763	—0.016446	+0.020863	—0.006211	—0.030728
1830	—0.013631	+0.000172	—0.006593	—0.016700	+0.042956	—0.007955	—0.025575
1840	—0.011458	+0.000239	—0.016844	—0.018081	+0.035628	—0.005227	—0.028549
1860	—0.012562	—0.002185	—0.026076	—0.010364	+0.038904	—0.006550	—0.031145
1880	—0.012433	—0.004284	—0.010790	—0.003637	+0.038543	—0.000581	—0.032358
1885	—0.012604	—0.005667	—0.024363	—0.007383	+0.039560	—0.000443	—0.027857

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years	$A_0^{(3)}$	$B_0^{(3)}$	$B_0^{(4)}$	$A_1^{(4)}$	$B_1^{(4)}$
1538	+ 0.006263	+ 0.009728	—	+ 0.204515	— 0.185687
1572	+ 0.004726	+ 0.013087	—	+ 0.019512	— 0.042398
1600	+ 0.004264	+ 0.001960	—	+ 0.035705	— 0.041210
1642	+ 0.004335	+ 0.019566	—	— 0.245348	+ 0.120000
1676	+ 0.000957	+ 0.014435	—	+ 0.079424	— 0.118363
1700	— 0.001970	+ 0.001662	—	+ 0.013895	— 0.101369
id.	—	—	—	—	—
1710	— 0.002055	+ 0.000248	—	+ 0.027658	— 0.005286
1720	— 0.005331	+ 0.006698	—	+ 0.082152	— 0.097625
1730	— 0.005783	+ 0.006873	—	— 0.054510	— 0.045622
1744	— 0.006551	+ 0.002791	—	+ 0.010207	— 0.016748
1756	— 0.009066	+ 0.001333	—	— 0.030747	— 0.030085
1760	—	—	—	—	—
1770	— 0.008399	+ 0.001694	—	+ 0.003841	— 0.036963
1780	— 0.004761	+ 0.007918	— 0.010504	+ 0.029979	— 0.059683
1787	— 0.008470	+ 0.001649	—	— 0.000563	— 0.039763
1800	— 0.009038	+ 0.001366	—	+ 0.038186	— 0.064707
1820	— 0.009141	+ 0.000736	— 0.002786	+ 0.027844	— 0.003377
1829	— 0.007248	— 0.001272	— 0.027377	— 0.003197	— 0.038405
1830	— 0.006552	+ 0.000488	— 0.038035	+ 0.022402	— 0.053291
1840	— 0.007175	— 0.001437	— 0.024255	— 0.000065	— 0.033215
1860	— 0.006473	— 0.002960	— 0.023236	— 0.005119	— 0.040117
1880	— 0.006812	— 0.003239	— 0.018963	— 0.014340	— 0.064692
1885	— 0.005492	— 0.003270	— 0.034395	— 0.011877	— 0.030597

years	$A_2^{(4)}$	$B_2^{(4)}$	$A_3^{(4)}$	$B_3^{(4)}$	$A_4^{(4)}$	$B_4^{(4)}$
1538	+ 0.001029	+ 0.093158	+ 0.012549	+ 0.013520	— 0.001437	— 0.002152
1572	+ 0.026228	+ 0.011724	+ 0.003476	— 0.006379	— 0.000892	— 0.000545
1600	+ 0.029696	— 0.021131	+ 0.001613	+ 0.003107	+ 0.002682	— 0.000855
1642	+ 0.031572	— 0.010102	— 0.001203	+ 0.000303	— 0.000905	— 0.006608
1676	+ 0.016411	— 0.008635	+ 0.005875	— 0.002058	— 0.001464	— 0.003795
1700	+ 0.020761	— 0.015794	+ 0.001076	+ 0.000859	+ 0.001026	— 0.000026
id.	—	—	—	—	—	—
1710	+ 0.024079	— 0.002749	+ 0.002019	+ 0.000911	+ 0.001692	+ 0.001005
1720	+ 0.045469	— 0.036582	+ 0.003260	+ 0.004734	+ 0.000681	+ 0.004087
1730	+ 0.060630	— 0.066835	+ 0.007790	+ 0.007283	+ 0.000818	— 0.001830
1744	+ 0.026539	— 0.007984	+ 0.004447	+ 0.001200	+ 0.001120	+ 0.000448
1756	+ 0.030131	+ 0.000145	+ 0.010959	+ 0.022307	— 0.000243	+ 0.001258
1760	—	—	—	—	—	—
1770	+ 0.024547	— 0.012719	+ 0.006874	+ 0.001956	+ 0.001042	+ 0.000290
1780	+ 0.010106	— 0.000496	+ 0.002016	+ 0.005909	+ 0.002973	+ 0.000549
1787	+ 0.019176	— 0.010822	+ 0.007866	+ 0.000485	+ 0.001379	+ 0.000171
1800	+ 0.016569	— 0.016654	+ 0.003247	— 0.001210	+ 0.001691	+ 0.000466
1820	+ 0.009462	— 0.035830	+ 0.006754	+ 0.004145	+ 0.000908	+ 0.001063
1829	+ 0.010851	— 0.015592	+ 0.003014	+ 0.006708	+ 0.001109	+ 0.000895
1830	+ 0.014876	— 0.016000	— 0.000062	+ 0.006909	+ 0.001109	+ 0.001442
1840	+ 0.007561	— 0.011004	+ 0.003133	+ 0.008165	+ 0.000311	+ 0.000149
1860	+ 0.017039	— 0.016568	+ 0.004219	+ 0.005815	+ 0.000561	+ 0.000336
1880	+ 0.014439	— 0.020615	+ 0.006755	+ 0.010284	+ 0.001027	— 0.000382
1885	+ 0.007147	— 0.019754	+ 0.005121	+ 0.006842	+ 0.000968	— 0.000849

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years	weight	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\beta_1^{(1)}$	$\beta_c - \beta_0$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$	$\beta_1^{(2)}$	$\beta_c - \beta_0$
1538	6	—	+ 0.071852	+ 38.442	— 9.103	—	+ 0.068652	+ 94.219	+ 5.832
1572	6	—	+ 0.084574	+ 47.687	— 14.459	—	+ 0.045493	+ 180.417	—
1600	8	—	+ 0.039203	+ 33.038	+ 3.393	—	+ 0.045418	+ 132.042	— 15.833
1642	6	—	+ 0.113758	+ 54.937	— 13.702	—	+ 0.041900	+ 132.179	— 5.024
1676	6	—	+ 0.101950	+ 47.706	— 2.382	—	+ 0.035594	+ 137.398	— 1.380
1700	8	—	+ 0.013474	+ 41.172	+ 6.697	—	+ 0.040695	+ 146.606	— 4.335
id.	5 1/4	+ 0.288251	+ 0.094297	+ 39.419	— 3.450	— 0.063404	+ 0.087422	+ 17.636	—
1710	7	—	+ 0.045608	+ 44.514	+ 4.499	—	+ 0.031053	+ 152.230	— 7.353
1720	5 1/4	—	+ 0.068197	+ 53.080	— 2.923	—	+ 0.046992	+ 142.531	+ 4.952
1730	6	—	+ 0.056559	+ 53.247	— 2.547	—	+ 0.049578	+ 149.429	+ 0.661
1744	7	—	+ 0.052866	+ 51.727	— 1.175	—	+ 0.048118	+ 148.353	+ 5.385
1756	5 1/4	—	+ 0.044486	+ 55.943	— 1.669	—	+ 0.094330	+ 86.798	—
1760	5 1/4	+ 0.347964	+ 0.068144	— 3.699	—	+ 0.024643	+ 0.144615	+ 130.619	—
1770	9	—	+ 0.056660	+ 56.721	— 0.345	—	+ 0.040245	+ 158.750	+ 1.764
1780	8	+ 0.319216	+ 0.065151	+ 43.507	+ 15.512	— 0.006100	+ 0.037464	+ 166.439	— 3.321
1787	9	—	+ 0.059247	+ 58.434	— 0.614	—	+ 0.041823	+ 157.895	+ 7.050
1800	8	—	+ 0.062995	+ 56.869	— 2.438	—	+ 0.021927	+ 152.279	+ 16.050
1820	21	+ 0.333810	+ 0.068341	+ 67.138	— 5.543	+ 0.005092	+ 0.030416	+ 179.823	— 6.278
1829	30	+ 0.320074	+ 0.066460	+ 64.747	— 2.123	+ 0.001210	+ 0.044542	+ 180.926	— 5.035
1830	21	+ 0.323477	+ 0.069773	+ 63.525	— 0.787	— 0.007708	+ 0.050678	+ 177.617	— 1.465
1840	27	+ 0.322334	+ 0.069368	+ 65.604	— 1.722	— 0.000354	+ 0.048108	+ 177.731	+ 1.027
1860	27	+ 0.325760	+ 0.066697	+ 63.894	— 2.276	— 0.001917	+ 0.047597	+ 177.839	+ 6.131
1880	30	+ 0.333923	+ 0.067807	+ 65.947	— 2.510	— 0.005507	+ 0.052069	+ 175.809	+ 13.374
1885	75	+ 0.315720	+ 0.065167	+ 67.618	— 1.411	+ 0.007906	+ 0.051466	+ 194.630	— 4.144

years	$\alpha_2^{(1)}$	$\beta_2^{(1)}$	$\beta_c - \beta_0$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$	$\beta_1^{(2)}$	$\beta_c - \beta_0$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\beta_c - \beta_0$
1538	+ 0.040316	— 67.944	— 8.123	—	+ 0.07470	+ 249.644	+ 8.441	+ 0.029373	+ 89.667	— 5.
1572	+ 0.037547	— 53.851	— 8.757	—	+ 0.055420	+ 276.393	— 9.061	+ 0.010880	+ 43.121	+ 41.
1600	+ 0.024531	— 30.690	— 20.834	—	— 0.001273	+ 322.559	— 47.612	+ 0.011534	+ 94.117	— 9.
1642	+ 0.044908	— 57.075	+ 22.177	—	— 0.115193	+ 254.002	+ 32.368	+ 0.011841	+ 68.603	+ 16.
1676	+ 0.022139	— 52.957	+ 31.518	—	— 0.055503	+ 262.994	+ 32.623	+ 0.018770	+ 79.543	+ 5.
1700	+ 0.017557	— 7.041	— 4.898	—	— 0.000055	+ 339.943	— 37.799	+ 0.021430	+ 91.908	— 6.
id.	+ 0.019256	+ 2.511	— 14.450	—	—	—	—	—	—	—
1710	+ 0.016683	— 3.052	— 4.928	—	— 0.005533	+ 84.863	—	+ 0.009500	+ 90.000	— 4.
1720	+ 0.012009	— 16.104	+ 12.082	—	— 0.055540	+ 68.387	—	+ 0.026702	+ 98.825	— 13.
1730	+ 0.013002	— 2.983	+ 2.920	—	— 0.055847	+ 44.031	—	+ 0.024589	+ 90.659	— 5.
1744	+ 0.018155	+ 3.256	+ 2.222	—	— 0.006663	+ 279.301	+ 34.810	+ 0.017546	+ 94.023	— 8.
1756	+ 0.015372	+ 9.146	+ 1.083	—	— 0.055558	+ 289.560	+ 27.814	+ 0.059613	+ 97.836	— 12.
1760	+ 0.064730	+ 52.030	—	—	—	—	—	—	—	—
1770	+ 0.009862	+ 14.642	+ 1.129	—	— 0.055745	+ 336.822	— 15.640	+ 0.027663	+ 83.221	+ 2.
1780	+ 0.014320	+ 30.854	— 11.125	— 0.016383	— 0.055642	+ 330.531	— 6.629	+ 0.020457	+ 85.458	+ 0.
1787	+ 0.010032	+ 18.843	+ 3.657	—	— 0.055508	+ 327.127	— 1.322	+ 0.028491	+ 83.490	+ 2.
1800	+ 0.012165	+ 22.446	+ 5.200	—	— 0.055512	+ 316.193	+ 13.148	+ 0.021883	+ 84.975	+ 0.
1820	+ 0.014637	+ 48.725	— 13.162	— 0.001792	+ 0.055655	+ 354.481	— 19.701	+ 0.016312	+ 80.729	+ 5.
1829	+ 0.012699	+ 42.177	— 3.051	— 0.018763	+ 0.055685	+ 331.152	+ 6.076	+ 0.031349	+ 84.286	+ 1.
1830	+ 0.013632	+ 44.638	— 5.116	— 0.005593	+ 0.055687	+ 338.755	— 1.255	+ 0.026783	+ 81.361	+ 4.
1840	+ 0.011461	+ 44.402	+ 0.922	— 0.016844	+ 0.055581	+ 335.475	+ 4.745	+ 0.029221	+ 83.848	+ 2.
1860	+ 0.012751	+ 49.933	+ 1.464	— 0.026076	+ 0.055511	+ 345.083	+ 0.576	+ 0.031827	+ 84.061	+ 1.
1880	+ 0.013150	+ 54.506	+ 4.808	— 0.010790	— 0.055513	+ 354.609	— 3.510	+ 0.032362	+ 89.485	— 3.
1885	+ 0.013819	+ 57.104	+ 4.189	— 0.024363	— 0.055523	+ 349.429	+ 3.029	+ 0.027860	+ 89.544	— 3.

years	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\beta_0 - \beta_0$	$\alpha_0^{(4)}$	$\alpha_1^{(4)}$	$\beta_1^{(4)}$	$\beta_0 - \beta_0$
1538	+ 0.012658	- 9'.885	- 3'.202	—	+ 0.276240	+ 227'.762	- 3'.205
1572	+ 0.013914	- 6.618	- 1.461	—	+ 0.046672	+ 204.712	+ 15.072
1600	+ 0.004691	- 21.768	+ 17.813	—	+ 0.054525	+ 220.906	- 5.053
1642	+ 0.020040	- 4.164	+ 6.395	—	+ 0.273121	+ 26.064	—
1676	+ 0.014467	- 1.264	+ 8.503	—	+ 0.142539	+ 213.863	- 8.680
1700	+ 0.002577	+ 16.616	- 5.842	—	+ 0.102316	+ 187.805	+ 14.009
id.	—	—	—	—	—	—	—
1710	+ 0.002070	+ 27.706	- 15.459	—	+ 0.028159	+ 259.180	- 58.770
1720	+ 0.008561	+ 12.839	+ 0.881	—	+ 0.127591	+ 220.081	- 21.075
1730	+ 0.008982	+ 13.359	+ 1.834	—	+ 0.071081	+ 129.929	+ 67.673
1744	+ 0.007121	+ 22.308	- 5.053	—	+ 0.019613	+ 211.360	- 15.723
1756	+ 0.009163	+ 27.212	- 8.190	—	+ 0.043017	+ 134.377	+ 59.575
1760	—	—	—	—	—	—	—
1770	+ 0.008568	+ 26.199	- 5.115	—	+ 0.037162	+ 185.932	+ 6.055
1780	+ 0.009241	+ 10.371	+ 12.186	- 0.010504	+ 0.066790	+ 206.671	- 16.088
1787	+ 0.008629	+ 26.328	- 2.740	—	+ 0.039766	+ 180.810	+ 8.790
1800	+ 0.009140	+ 27.135	- 1.632	—	+ 0.075134	+ 210.548	- 22.773
1820	+ 0.011365	+ 17.847	+ 10.602	- 0.002786	+ 0.028048	+ 263.085	- 78.118
1829	+ 0.007359	+ 33.318	- 3.544	- 0.027377	+ 0.038536	+ 175.241	+ 8.463
1830	+ 0.008170	+ 17.774	+ 12.148	- 0.038035	+ 0.057809	+ 202.801	- 19.238
1840	+ 0.007318	+ 33.775	- 2.380	- 0.024255	+ 0.033215	+ 179.888	+ 2.271
1860	+ 0.007117	+ 38.191	- 3.851	- 0.023236	+ 0.040442	+ 172.728	+ 6.624
1880	+ 0.007543	+ 38.477	- 1.191	- 0.018963	+ 0.066261	+ 167.502	+ 9.042
1885	+ 0.006392	+ 40.257	- 2.234	- 0.034395	+ 0.032819	+ 158.784	+ 17.058

years	$\alpha_3^{(4)}$	$\beta_3^{(4)}$	$\beta_0 - \beta_0$	$\alpha_3^{(4)}$	$\beta_3^{(4)}$	$\beta_0 - \beta_0$	$\alpha_4^{(4)}$	$\beta_4^{(4)}$	$\beta_0 - \beta_0$
1538	+ 0.093162	+ 179'.683	—	+ 0.018444	+ 345'.713	- 23'.235	+ 0.002587	+ 36'.567	+ 13'.212
1572	+ 0.028729	+ 147.042	+ 6'.375	+ 0.007264	+ 309.529	+ 15.982	+ 0.001045	+ 30.356	+ 21.435
1600	+ 0.036447	+ 117.282	+ 18.426	+ 0.003501	+ 350.855	- 22.847	+ 0.002814	+ 63.077	—
1642	+ 0.033148	+ 126.128	+ 4.594	+ 0.001241	+ 25.288	—	+ 0.006670	+ 43.051	+ 12.884
1676	+ 0.018544	+ 121.124	+ 5.561	+ 0.006225	+ 323.565	+ 11.221	+ 0.004068	+ 39.726	+ 18.222
1700	+ 0.026086	+ 116.369	+ 7.467	+ 0.001377	+ 342.867	- 5.940	+ 0.001026	+ 67.137	- 7.768
id.	—	—	—	—	—	—	—	—	—
1710	+ 0.024236	+ 131.743	- 9.094	+ 0.002215	+ 338.095	- 0.376	+ 0.001968	+ 75.177	- 15.216
1720	+ 0.058356	+ 115.590	+ 5.872	+ 0.005748	+ 348.482	- 9.771	+ 0.004143	+ 87.635	—
1730	+ 0.090238	+ 111.105	+ 9.168	+ 0.010664	+ 344.358	- 4.755	+ 0.002004	+ 51.021	+ 10.124
1744	+ 0.027713	+ 126.628	- 8.016	+ 0.004606	+ 335.034	+ 5.817	+ 0.001206	+ 72.952	- 10.978
1756	+ 0.030131	+ 179.862	—	+ 0.024853	+ 351.279	- 9.357	+ 0.001281	+ 92.733	—
1760	—	—	—	—	—	—	—	—	—
1770	+ 0.027646	+ 121.354	- 5.828	+ 0.007147	+ 335.295	- 7.875	+ 0.001081	+ 71.378	- 7.865
1780	+ 0.010118	+ 133.595	- 19.257	+ 0.006241	+ 353.720	- 9.658	+ 0.003023	+ 70.115	- 6.010
1787	+ 0.022019	+ 120.280	- 6.773	+ 0.007881	+ 331.176	+ 13.511	+ 0.001787	+ 69.267	- 4.748
1800	+ 0.023492	+ 112.573	- 0.609	+ 0.003465	+ 323.187	+ 22.659	+ 0.001754	+ 71.352	- 6.063
1820	+ 0.037058	+ 97.396	+ 12.194	+ 0.004210	+ 356.644	- 9.014	+ 0.001398	+ 79.873	- 13.400
1829	+ 0.018996	+ 107.418	+ 1.103	+ 0.007354	+ 351.935	- 3.502	+ 0.001425	+ 77.226	- 10.220
1830	+ 0.021847	+ 111.457	- 3.055	+ 0.006909	+ 0.171	- 11.649	+ 0.001819	+ 80.609	- 13.544
1840	+ 0.013351	+ 107.247	- 0.032	+ 0.008745	+ 353.003	- 3.589	+ 0.000345	+ 73.900	- 6.243
1860	+ 0.023766	+ 112.901	- 8.060	+ 0.007184	+ 348.012	+ 3.185	+ 0.000654	+ 75.230	- 6.389
1880	+ 0.025168	+ 107.504	- 5.038	+ 0.012304	+ 348.900	+ 4.081	+ 0.001096	+ 62.399	+ 7.626
1885	+ 0.021007	+ 99.945	+ 1.928	+ 0.008546	+ 347.729	+ 5.698	+ 0.001288	+ 57.187	+ 13.134

below. The $\gamma_n^{(i)}$ values correspond to the year 1800.

$\alpha_0^{(1)} = +0.322946$	$\gamma_1^{(1)} = 39.307$	$m_1^{(1)} = +0.11438$
$\alpha_1^{(1)} = +0.067055$	$\gamma_2^{(1)} = 168.333$	$m_2^{(1)} = +0.26062$
$\alpha_2^{(1)} = +0.001506$	$\gamma_3^{(1)} = 27.546$	$m_3^{(1)} = +0.39585$
$\alpha_3^{(1)} = +0.047815$	$\gamma_4^{(1)} = 329.345$	$m_4^{(1)} = +0.27197$
$\alpha_4^{(1)} = +0.013244$	$\gamma_5^{(1)} = 85.638$	$m_5^{(1)} = +0.00540$
$\alpha_5^{(1)} = -0.017527$	$\gamma_6^{(1)} = 25.503$	$m_6^{(1)} = +0.14729$
$\alpha_6^{(1)} = +0.039018$	$\gamma_7^{(1)} = 157.775$	$m_7^{(1)} = -0.14039$
$\alpha_7^{(1)} = +0.028373$	$\gamma_8^{(1)} = 111.564$	$m_8^{(1)} = -0.11872$
$\alpha_8^{(1)} = +0.007474$	$\gamma_9^{(1)} = 545.346$	$m_9^{(1)} = +0.08919$
$\alpha_9^{(1)} = -0.026447$	$\gamma_{10}^{(1)} = 65.239$	$m_{10}^{(1)} = +0.05920$
$\alpha_{10}^{(1)} = +0.040680$		
$\alpha_{11}^{(1)} = +0.022249$		
$\alpha_{12}^{(1)} = +0.008200$		
$\alpha_{13}^{(1)} = +0.001155$		

By calculating these numbers we have given an unequal weight to the different years and we estimated this weight on the basis of the number of parallels entering in the condition equations. The weight of the incomplete observations of longitude decreased by a ratio of 3/4, based on the number of meridians shown on the charts. I allowed myself to exclude a small number of observations which exhibited extraordinary discrepancies. The $\beta_n^{(i)}$ s thus eliminated are followed by a question mark. The $\alpha_n^{(i)}$ coefficients were calculated on the basis of the most accurate data of the present century, where the intensity of the magnetic force is known. Furthermore, I found that even if I included ancient observations, the values adopted do not change appreciably.

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The errors resulting from these formulas in the $\beta_n^{(i)}$ angles are shown next to these in the $\beta_c - \beta_0$ columns. If we inspect these numbers, which are often quite large, we cannot say that our corrections gave completely satisfactory results. Yet, most of the $\gamma_n^{(i)}$ functions move in

a positive direction, like the lower order terms. Only two of them [the $y_l^{(4)}$ and $y_n^{(4)}$ functions] exhibit a negative direction. Furthermore, I still have some reservations about this, because these functions are certainly very incorrectly determined, as may be concluded from the enormous values acquired by the $\beta_c - \beta_o$ deviations.

In summary, we arrive at this somewhat discouraging result, that from the seven new functions included in our second approximation, only one or two seem to be determined with sufficient accuracy: these are the third order functions [$y_1^{(3)}$ and $y_3^{(3)}$].

Without any hesitation, we ascribe this result to the imprecisions in Hansteen's charts. As a matter of fact, we have based all of our calculations on the cartographic representations, which we sketched by hand, according to documents which are often highly precarious, and by allowing ourselves to be guided by a sense of continuity. This was obviously a very unlogical way to correct the observation errors, but it greatly simplified the calculations.

Now that we have approximated values of the unknowns, we can advantageously substitute this for a more rigorous method of calculation. This work must be preceded by a very thorough comparison of the theory with the observations: each observation of the declination, inclination or intensity, made at any period of time, will result in a difference: observation minus theory, and the set of corrections thus found will make it possible to obtain the most probable values of the unknowns and to thus reach highest level of perfection of which the theory seems capable.

10.

Although we must not consider the numerical part of our work to be completed, the results obtained already make it possible to formulate certain statements concerning the general form of the secular inequalities.

Our calculations bring us to represent the function of the magnetic forces at a single given place with a series of terms having the expression:

$$P_n^{(0)} a_n^{(0)} \cos (n\pi + n\gamma_n^{(0)} + nm_n^{(0)} t),$$

where the term t stands for the cosine sign and is the only variable element. A similar form is convenient for representing the three components of force, except for the component perpendicular to the meridian, the cosine must be replaced by a sine (no. 6).

The expression for the horizontal component perpendicular to the meridian does not contain a constant term: for a long enough time factor, its mean value is zero, and the value of the component oscillates about the zero value. This must also be true for the magnetic declination, at least approximatively.

The other two components of force, which are within the meridian plane, contain in addition to the variable part, a constant part about which the horizontal and vertical intensities oscillate.

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Besides these constant parts, each of the components contains a series of periodical terms with a more or less long cycle.

If we continue to let $m_n^{(i)}$ be the average yearly longitudinal displacement of argument $\beta_n^{(i)}$, the revolution time of the term $Y_n^{(i)}$ will be $\frac{2\pi}{m_n^{(i)}}$ and this term will generate a corresponding cycle term $\frac{2\pi}{nm_n^{(i)}}$, in the expressions of the components of force.

I find in particular, for the main terms for which calculations already made make it possible to determine the yearly motion $m_n^{(i)}$ with some accuracy, the following cycle values:

Term	Cycle	
	β, X, Y, Z	
$Y_1^{(1)}$	314-Yrs	3147 Yrs
$Y_1^{(2)}$	1381	1381
$Y_2^{(2)}$	922	454
$Y_1^{(3)}$	1322	1324
$Y_3^{(3)}$	2422	815

For the elements provided directly by observation, the theory does not indicate any simple expression, and it is easily conceivable that there are no determined cycles for different places, as we often thought we noticed.

The magnetic axis of the earth is the direction of the main magnetic moment of the earth. With this, we find that the earth's magnetic axis describes a circular cone about the rotating axis, with an angular distance of $23^\circ 28'$ in 3147 years. According to the results of our calculations, the magnitude of this moment is invariable and equal to $R^3 \sqrt{a_0^{(1)2} + a_1^{(1)2}}$, or in an absolute value, to the number 852640 followed by 20 figures, if centimeters

grams, seconds for average time are adopted for units.

The displacements of the earth's magnetic poles are not as simple. These points are characterized each instant by the condition $X = 0$, $Y = 0$, where X and Y are in the form mentioned earlier. These are two transcendental equations in μ and ϖ ; after eliminating time, they give an equation between μ and ϖ , which defines a cone whose outline with the sphere is the trajectory described by the magnetic poles over the earth's surface.

If we limit ourselves to the first order terms only, the magnetic poles would become indistinguishable from the points where the magnetic axis passes through the terrestrial surface and would move with them. Moreover, the higher order terms essentially change the nature of the trajectory. If we again adopt the uniform motion for the first approximation, we obtain equations with finite differences which we satisfy by introducing the periodical terms in μ and ϖ . We may thus account for the generation of these yawed curves which various authors have obtained using a completely different method (Astronomische Nachrichten, No.s 3254 and 3299.)*

The earth's magnetic equator satisfies for each of these points the conditions $Z = 0$, where Z is a known function of μ and ϖ . This is a complicated equation which would seem embarrassing to discuss. The points of intersection with the terrestrial equator ($\mu = 0$), are easier to determine. The longitudes of the ascending and descending nodes of the magnetic equator on the terrestrial equator are given by the following equation:

*We read in most treatises that the earth's magnetic poles describe a complete circumference in about 600 years: this result is not accurate: we have found that it is the second and third order terms which make it seem like a cycle of about 600 years.

$$\begin{aligned}
 0 = & -\alpha_0^{(2)} + 2\alpha_1^{(1)} \cos (\pi + \gamma_1^{(1)} + m_1^{(1)}t) \\
 & + 3\alpha_2^{(2)} \cos (2\pi + 2\gamma_2^{(2)} + 2m_2^{(2)}t) \\
 & - \frac{1}{5}\alpha_1^{(3)} \cos (\pi + \gamma_1^{(3)} + m_1^{(3)}t) \\
 & + 4\alpha_3^{(3)} \cos (3\pi + 3\gamma_3^{(3)} + 3m_3^{(3)}t) \\
 & + \frac{3}{7}\alpha_0^{(4)} \\
 & - \frac{5}{7}\alpha_2^{(4)} \cos (2\pi + 2\gamma_2^{(4)} + 2m_2^{(4)}t) \\
 & + 5\alpha_4^{(4)} \cos (4\pi + 4\gamma_4^{(4)} + 4m_4^{(4)}t).
 \end{aligned}$$

This is an equation of the eighth degree in $\cos \pi$ with real coefficients. For each determined value of t , the equation with eight roots including one or several pairs may be real.

11.

We had nothing to show in these pages for a theory with a numerical perfection, which can be achieved only by operating step by step. Our objective was more modest.

This was to show a method by which we can hope to penetrate into a difficult subject and to show the results which we may hopefully obtain through a systematic application of calculus: this is the task which we were allowed to undertake and which we tried to fulfill.

However, in the results already obtained, we will find solid ground upon which we may base all of our subsequent investigations and will enable us to calculate with the precision we want the higher order terms which we missed during the first test.

A comparison of the theory with the observations will give us a better idea of the direction to take in such a

revised calculation. It is for this purpose only that we want to assess the still imperfect results of a comparison of the theory with observations.

For our comparison term, we have selected the declination and the inclination for the year 1700. For this date, we have calculated the magnetic force and its direction based on the numerical constants obtained in the preceding pages.

We will succinctly indicate the steps taken in the calculation of the magnetic force and its direction.

The modules $\alpha_n^{(i)}$ of the functions $\gamma_n^{(i)}$ and their arguments $\beta_n^{(i)}$ are first calculated according to the formulas of no. 9, by using the numerical values given in this chapter.

As these transcendental quantities are assumed to be known, we will easily pass to the constants $A_n^{(i)}$ and $B_n^{(i)}$, using the formulas of no. 5, after which we may easily obtain the coefficients $L_n, L'_n, M_n, M'_n, N_n, N'_n$, which appear in the expansions of the forces as a function of the longitude multiples.. These coefficients, and the components of force X, Y and Z themselves, are given by the formulas of no. 6.

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Now all we have to do is calculate the elements we are looking for: S, i and δ which are obtained using a well-known system described in no. 1.

The tables below contain the intensity and the direction of the earth's magnetic attraction in the year 1600, calculated on the basis of a provisional equation for the points of intersection of thirteen parallels equally spaced with twelve meridians.

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Total force												
0	$\pi = 0^\circ$	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
0	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919	0.5919
15	0.5728	0.5749	0.5822	0.5904	0.5949	0.5933	0.5921	0.5955	0.6005	0.6001	0.5909	0.5789
30	0.5464	0.5642	0.5729	0.5665	0.5549	0.5459	0.5565	0.5902	0.6227	0.6205	0.5828	0.5480
45	0.5098	0.5259	0.5223	0.5020	0.4813	0.4673	0.4819	0.5433	0.6149	0.6284	0.5743	0.5169
60	0.4575	0.4581	0.4344	0.4063	0.3912	0.3857	0.4023	0.4686	0.5604	0.5893	0.5356	0.4721
75	0.3820	0.3702	0.3544	0.3406	0.3363	0.3387	0.3474	0.3908	0.4761	0.5114	0.4679	0.4071
90	0.3172	0.3034	0.3156	0.3287	0.3409	0.3589	0.3585	0.3725	0.4055	0.4366	0.4071	0.3516
105	0.2907	0.2784	0.3109	0.3524	0.3852	0.4123	0.4009	0.3770	0.4011	0.4328	0.4091	0.3429
120	0.3172	0.3039	0.3503	0.4178	0.4741	0.5003	0.4919	0.4597	0.4755	0.5078	0.4799	0.3933
135	0.3853	0.3655	0.4097	0.4835	0.5525	0.5894	0.5796	0.5561	0.5651	0.5833	0.5501	0.4642
150	0.4763	0.4600	0.4936	0.5553	0.6153	0.6494	0.6507	0.6402	0.6397	0.6367	0.6016	0.5357
165	0.5756	0.5685	0.5846	0.6159	0.6489	0.6717	0.6804	0.6793	0.6729	0.6595	0.6352	0.6027
180	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526	0.6526

Inclination												
0	$\pi = 0^\circ$	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
0°	+ 86°13'	+ 86°13'	+ 86°13'	+ 86°13'	+ 86°13'	+ 86°13'	- 86°13'	+ 86°13'	+ 86°13'	+ 86°13'	+ 86°13'	+ 86°13'
15	+ 84 15	+ 83 44	+ 82 26	+ 80 18	+ 78 29	+ 78 3	- 79 3	+ 81 25	+ 85 15	+ 89 27	+ 87 22	+ 85 19
30	+ 78 36	+ 77 41	+ 74 40	+ 71 2	+ 67 51	+ 66 23	- 67 39	+ 73 1	+ 80 34	+ 85 24	+ 82 16	+ 79 24
45	+ 70 52	+ 68 11	+ 62 29	+ 57 4	+ 53 27	+ 50 22	- 51 50	+ 60 56	+ 71 58	+ 77 19	+ 74 47	+ 71 39
60	+ 58 22	+ 54 1	+ 44 40	+ 37 3	+ 32 54	+ 27 31	- 29 5	+ 44 4	+ 59 13	+ 64 53	+ 62 55	+ 59 51
75	+ 42 35	+ 34 46	+ 18 36	+ 8 3	+ 4 36	+ 1 38	- 0 46	+ 21 47	+ 39 45	+ 44 21	+ 43 9	+ 42 42
90	+ 19 48	+ 9 16	- 11 25	- 22 23	- 24 32	- 29 39	- 26 46	- 5 36	+ 11 52	+ 14 24	+ 13 47	+ 17 2
105	- 4 23	- 13 42	- 33 11	- 42 52	- 44 52	- 48 17	- 45 4	- 29 5	- 18 13	- 18 50	- 18 29	- 11 13
120	- 31 22	- 36 23	- 49 18	- 57 38	- 60 36	- 62 43	- 59 23	- 50 7	- 45 52	- 47 54	- 46 47	- 38 56
135	- 47 59	- 49 27	- 57 55	- 65 39	- 70 27	- 72 32	- 59 24	- 64 11	- 63 17	- 65 20	- 63 4	- 55 6
150	- 60 35	- 60 19	- 64 53	- 71 17	- 77 20	- 80 55	- 73 13	- 76 8	- 75 36	- 75 53	- 72 25	- 65 50
165	- 71 12	- 70 47	- 72 52	- 76 43	- 81 25	- 86 7	- 88 51	- 86 25	- 84 22	- 81 48	- 78 2	- 74 0
180	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18	- 81 18

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Declination

θ	$\varpi = 0^\circ$	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
0°	+115'13"	+85'13"	+55'13"	+25'13"	-4'47"	-34'47"	-64'47"	-94'47"	-124'47"	-154'47"	+175'13"	+145'13"
15°	+27 9	+25 37	+25 38	+18 33	+7 1	-8 13	-27 43	-48 39	-68 52	-96 44	+66 48	+41 20
30°	+10 47	+13 1	+16 47	+14 7	+7 32	-2 44	-17 47	-33 47	-41 10	-2 12	+30 55	+21 1
45°	+7 11	+11 22	+14 9	+10 53	+6 31	-0 38	-13 52	-26 51	-27 5	-2 5	+17 30	+13 19
60°	+6 3	+11 40	+13 31	+9 5	+5 34	+0 29	-11 53	-22 38	-19 21	-2 28	+8 58	+7 43
75°	+5 28	+12 49	+15 34	+8 11	+5 5	+1 8	-11 32	-20 3	-13 46	-2 17	+3 20	+3 14
90°	+4 2	+14 3	+15 52	+9 8	+5 41	+1 23	-11 43	-17 29	-9 30	-1 58	-0 41	-1 13
105°	+1 23	+14 55	+19 18	+12 40	+7 49	+1 18	-10 14	-14 43	-5 56	-1 46	-4 30	-6 11
120°	-5 41	+14 17	+23 11	+19 15	+12 20	+1 42	-12 9	-11 57	-3 26	-2 31	-9 31	-12 15
135°	-6 45	+12 0	+25 51	+27 59	+20 30	+3 54	-11 37	-10 31	-3 28	-6 5	-17 0	-18 40
150°	-9 40	+10 8	+27 27	+37 16	+35 48	+13 53	-11 4	-12 57	-9 52	-17 26	-28 7	-24 42
165°	-11 6	+10 23	+31 34	+50 48	+66 59	+75 25	-5 12	-45 47	-48 51	-50 45	-44 44	-30 26
180°	-11 1	+18 59	+48 59	+78 59	+108 59	+138 59	+168 59	-161 1	-131 1	-101 1	-71 1	-41 1

Comparison of the theory with observations

θ	$\varpi = 0^\circ$	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
30°	+19'.28	+12'.42	-0'.01	-3'.69	+7'.54	+17'.27	+22'.22	-26'.79	-101'.16	-39'.20	+9'.32	+15'.72
45°	+13.99	+9.16	+1.25	-1.61	+3.32	+12.37	+16.13	-21.85	-47.08	-24.69	+2.50	+12.61
60°	+8.75	+6.46	-1.67	-1.11	+5.96	+11.48	+8.12	-12.63	-14.65	-16.86	+0.36	+9.53
75°	+6.67	+5.31	-3.03	-1.82	+7.08	+11.14	+5.47	-9.05	-9.77	-8.93	+0.14	+6.93
90°	+6.63	+6.25	-5.44	-2.87	+7.09	+10.39	-1.72	-11.48	-6.50	-3.96	-0.89	+7.18
105°	+8.09	+8.13	-3.90	-3.04	+6.12	+7.79	-4.23	-14.72	-6.43	-3.60	-0.51	+7.12
120°	+6.71	+9.28	-0.42	-1.76	+5.14	+5.69	-5.65	-12.95	-7.44	-3.51	+0.49	+6.94
135°	+6.95	+10.20	+4.26	+0.09	+7.10	+6.90	-3.61	-7.52	-3.48	-2.38	-3.40	+6.64

The last table contains the results of the comparison of the theory with observations. In it we show the differences between the calculated and observed declinations, in direction $\delta_c - \delta_o$. The pattern of the differences shows that they are essentially due to the terms multiplied by the arguments of the expression $2\varpi + 2\gamma_2^{(1)} + 2m_2^{(1)}t$.

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A quick calculation has given the following results for the arguments of the terms dependent upon ϖ and 2ϖ , in the third approximation:

$$\begin{array}{ll} \beta_1^{(1)} = 37'.249, & - \\ \beta_1^{(2)} = 121'.805, & \beta_2^{(2)} = 329'.893, \\ \beta_1^{(3)} = 266'.631, & \beta_2^{(3)} = 290'.256, \\ \beta_1^{(4)} = 53'.798, & \beta_2^{(4)} = 283'.765; \end{array}$$

from which we would conclude that the motion of all fourth order arguments is fairly rapid, in a positive direction.

There are a certain number of inclination observations /28
at the turn of the XVIIth centry, made by scientists in Europe and by English voyagers looking for north-west and north-east passages. We are reporting these observations here, although may be considered only very rough figures.

We have made important corrections to the inclinations of Baffin, one due to a collimation error, and the other to an incorrectly balanced suspension. The rest of the observations were left the same.

We may find it of some interest to compare them with the theory, and we can do this appoximately by using our inclination table. We will find that the theory represents the observations to the nearest 1 or 2°, and as the inclination has diminished by about 6° over the years, we see that the theory already accounts for most of the variation. To have a better appreciation of the value of this result, let us remember that the oldest inclination observations, which served to define the arbitraries of the theory, were borrowed from Hansteen's chart for 1780.

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Observations Of The Magnetic Inclination In The XVIth And XVIIth Centuries

a. Lieux.	b Latitude 90°-0	c Longitude E	d Dates.	e Inclinaison i	f Autorités.
1 Hornsound	+ 76°.55'	15°.30'	1613 juin 14	+ 86°. 0'	Baffin.
2 Mer glaciale	+ 75.22	41. 0'	1608 juin 19	+ 89.30	Hudson.
"	+ 74.35	44.30'	1608 juin 22	+ 85. 0	Hudson.
"	+ 74.23	26. 0'	1608 juin 7	+ 86. 0	Hudson.
3 Swarte-Cliff (Nouv. Zem ^e)	+ 72.12	51. 0	1608 juin 29	+ 84. 0	Hudson.
4 Cap-Nord	+ 71. 0	25.50'	1608 ju. 243	+ 84.30	Hudson.
5 Côte de la Norvège . . .	+ 69.40	17.450	1608 mai 26	+ 84. 0	Hudson.
"	+ 67. 0	13. 5	1608 mai 22	+ 82. 0	Hudson.
6 Baie de Baffin	+ 65.45	303. 0	1613 mai 23	+ 82. 0	Baffin.
7 Côte de la Norvège . . .	+ 64.52	10.25	1608 m. 255	+ 81. 0	Hudson.
8 Frobisher Strait	+ 61.40	295. 0	1602 ju. 247	+ 83.30	Weymouth.
9 Près Shetland	+ 61.11	357.40	1607 mar 243	+ 79. 0	Hudson.
10 Uranibourg	+ 55.54	12.42	1583-1585	+ 73.25	Tycho-Brahé.
11 Londres	+ 51.31	359.54	1576	+ 71.50	R. Norman.
"	"	"	1600	+ 72. 0	Gilbert.
"	"	"	1613	+ 72.30	Ridley.
"	"	"	1676	+ 73.30	Bond.
12 Rouen	+ 49.26	1. 5	1639 ?	+ 72. 0	P. Grandami.
13 Paris	+ 48.50	2. 22	1671 26	+ 75. 0	Richer.
14 Kauffbeuren	+ 47.53	10.50'	1607 nov. 28	+ 70. 0	Képler.
15 Tours	+ 47.24	359.10	1639 ?	+ 70. 0	P. Grandami.
16 Ferrare	+ 44.51	11.50'	1629	+ 62. 0	Cabréus.
17 Rome	+ 41.54	12.20'	1639 ?	+ 65.50	P. Kircher.
18 Naples	+ 40.52	14.10'	1639 ?	+ 59.15	P. Kircher.
19 Lisbonne	+ 40. 0	352.10'	1639 ?	+ 62.45	P. Kircher.
20 Messine	+ 38.11	15.55	1639 ?	+ 59.15	P. Kircher.
21 Palerme	+ 38. 8	13.25	1639 ?	+ 59.15	P. Kircher.
22 Malte	+ 35.54	15.50'	1639 ?	+ 59.15	P. Kircher.

Key: a-Place; b-Latitude; c-Longitude;
d-Dates; e-Inclination; f-Authorities;
1-Hornsound; 2-Ice Sea; 3-Swarte-Cliff
(New Zem); 4-Cape-North; 5-Coast of
Norway; 6-Baffin Bay; 7-Coast of Norway;
8-Frobisher Strait; 9-Near Shetland;
10-Uranibourg; 11-London; 12-Rouen;
13-Paris; 14-Dauffbeuren; 15-Tours;
16-Ferrare; 17-Rome; 18-Naples; 19-Lisbon;
20-Messina; 21-Palerma; 22-Malta.
23-June; 24-May; 25-July; 26-November.

For the inclination in the XVIth and XVIIth
consult the following publications:

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- Hansteen, Untersuchungen über den Magnetismus der Erde, t. I, p. 38.
- Fortunio Affaitati, De peculiari magnetis ad polum descensu etc. (Physicae et astronomicae considerationes, Venet, 1549).
- R. Norman, The New Attractive, London 1580.
- Cabaeus, Philosophia magnetica, Ferrariae 1629.
- Kircher, Magnes, sive de Arte magnetica, p. 401, 424 (1641 edition).
- Jacobus Grandamicus, Noua demonstratio Immobiliatis Terroe, cap. 4, sect.1, La Flèche 1645.

- Tycho-Brahé's and Képler's observations were made from the position of the point of convergence of the northern light beams observed by these astronomers.

- Tycho-Brahé's meteorological log-book recorded at Uranibourg from 1582-1597, published in Copenhagen in 1876 by Paul la Cour, p. XLV.

-Marian, Treatise On The Northern Lights, p. 130, 188 (Continuation of Mémoires de l'Académie Royale des Sciences, 1731).

12.

The objective of our work was primarily to offer a numerical theory of the earth's magnetic attraction, applicable to any time, and we have arrived at a solution to this problem by approximating the truth.

This solution. although it already appeared in an excessively simply analytical form, was nevertheless of a purely empirical nature. And this type of solution is not satisfactory to the scientific spirit. The true objective of the theory is to trace back to the physical

causes and to thus give the formulas their theoretical justification, by starting with the principles of physics or mechanics alone.

From this standpoint, the approximated solution found by inductive reasoning, may be considered an approximatively known integral of the problem. To arrive at this objective, it is necessary to deduce from it the forces exerted at each moment. This will make it possible to express the differential equation which the question is dependent upon, for the hypothesis adopted.

In the present chapter, we will take up this difficult problem, by slightly exceeding the boundaries of precision which we scrupulously tried to observe until now.

Let us state first that any complete theory of the earth's magnetic attraction seems possible only if the effects observed come from a constant cause which always acts in the same way. If, for example, the changes were due to a slow cooling of the earth, or to the transfer of matter to unknown depths, how can they be as constant as they are?

Let us therefore assume that the changes are due to a constant and always uniform cause, and above all to a continuous remagnetization of the earth, which is assumed to be a continuous magnet. Now we will ask ourselves: what could a magnetic field be, at each instant, which would produce the remagnetization observed?

Let us assume there is a sphere magnetized by any, but variable, mode. Let V be the sphere's potential for an outside point at any moment. Also let Ω be the potential at the same point due to the induced magnetization

developed within the sphere, per unit of time, by a system of electrical currents or by any other foreign cause. By borrowing a mode of language specific to mechanics, we will call this the virtual potential of the sphere. Once this is established, the total or instantaneous potential V will be related to the virtual potential by a relationship expressed:

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$$\frac{dV}{dt} = \Omega$$

where Ω is estimated to contain a numerical factor which is equal to the quantity of the continuous magnetism developed within the sphere per unit of time.

We are proposing to determine this virtual magnetization as a function of the instantaneous magnetization, so as to satisfy the conditions created by the earth.

We will consider an especially simple case, that of a homogeneous spherical layer, or of a sphere composed of concentric and homogeneous layers.

The potential Ω of the magnetization induced in the sphere is assumed to be caused by an outside magnetic field V' which we are trying to determine. Poisson's theory told us to determine Ω as a function of V' , on the assumption that Ω is expanded into a series of spherical functions, where each $\Omega^{(i)}$ term of order i is directly proportional to a term of the same $V'^{(i)}$ order in the inductor field V' , so that we may express:

$$\Omega^{(i)} = N_i V'^{(i)},$$

by letting N_i be the factor of proportionality.

We are therefore assuming that the continuous magnetism follows the same laws as a temporary magnetism aroused by induction. The time factor may be estimated to be contained in the factor N_i , which also contains the outside and inside half-diameters a, b of the hollow sphere, the order i of the function $V^{(i)}$, and the coefficient of magnetization induced k in the middle.

An analogous expression is obviously appropriate for the case of a sphere composed of concentric and homogeneous layers, or even for a sphere where the coefficient of magnetization is a function of the vector radius alone. We simply have to replace the expression above by a sum, where each element corresponds to an isolated layer, or by an integral.

By letting N'_i be the factor of proportionality and by setting:

$$\Omega = \Omega^{(0)} + \Omega^{(1)} + \Omega^{(2)} + \text{etc.},$$

we will always have the following relationship:

$$\Omega^{(i)} = N V^{(i)}$$

Once this is established, the differential equation which defines the total potential of the earth V at each instant, as a function of time, will be:

$$\frac{dV}{dt} = N_0 V^{(0)} + N_1 V^{(1)} + N_2 V^{(2)} + \text{etc.},$$

where the numbers N_0, N_1, N_2 , vary from one term to another in the series.

If the second member is zero, this equation becomes:

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V = Arbitrary function of μ , ϖ ;

the arbitrary function being determined by the initial state.

To solve the proposed equation in the case where the second member is not zero, we expand V into a series of Laplace functions and we will apply the method of variation of arbitrary constants to it. We will therefore set:

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \text{etc.}; \quad V^{(i)} = V_0^{(i)} + V_1^{(i)} + \dots + V_i^{(i)};$$

$$V_n^{(i)} = (V_1 - \mu^2)^n \cdot \frac{d^n X_i}{d\mu^n} \cdot \alpha_n^{(i)} \cos n(\varpi + \beta_n^{(i)}),$$

and the analogous expressions for the $V_n^{(i)}$, which are deduced from the first ones by replacing $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ with $\alpha_n^{(i)}$ and $\beta_n^{(i)}$.

If we vary the $\alpha_n^{(i)}$ s and $\beta_n^{(i)}$ s, and identify the $\cos n\varpi$ and $\sin n\varpi$ coefficients of the two members, we will have a system of conditions [we have omitted the common factor $(V_1 - \mu^2)^n \frac{d^n X_i}{d\mu^n}$]:

$$\cos n\beta_n^{(i)} \frac{d\alpha_n^{(i)}}{dt} - n\alpha_n^{(i)} \sin n\beta_n^{(i)} \frac{d\beta_n^{(i)}}{dt} = N_i \alpha_n^{(i)} \cos n\beta_n^{(i)},$$

$$\sin n\beta_n^{(i)} \frac{d\alpha_n^{(i)}}{dt} + n\alpha_n^{(i)} \cos n\beta_n^{(i)} \frac{d\beta_n^{(i)}}{dt} = N_i \alpha_n^{(i)} \sin n\beta_n^{(i)};$$

from which we may promptly deduce:

$$\frac{d\alpha_n^{(i)}}{dt} = N_i \alpha_n^{(i)} \cos n(\beta_n^{(i)} - \beta_n^{(i)}); \quad \frac{d\beta_n^{(i)}}{dt} = \frac{N_i \alpha_n^{(i)}}{n\alpha_n^{(i)}} \sin n(\beta_n^{(i)} - \beta_n^{(i)}).$$

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If $\alpha_n^{(i)}$ does not contain a secular term, as shown in the observations, it is necessary for $n(\beta_n^{(i)} - \beta_n^{(i)}) = (2k + 1)\frac{\pi}{2}$, by letting k be any whole number. On this assumption, the second equation is expressed:

$$\frac{d\beta_n^{(i)}}{dt} = \pm \frac{N_i \alpha_n^{(i)}}{n\alpha_n^{(i)}}$$

If it is integrated, it becomes:

$$\beta_n^{(i)} = \pm \frac{N_i \alpha_n^{(i)}}{n\alpha_n^{(i)}} \cdot t + \gamma_n^{(i)},$$

and by letting $\gamma_n^{(i)}$ be an integration constant.

The observations show that the factor t is positive, and since N_i is negative, it is therefore necessary to use the lower sign, which shows that k is an odd number. It follows that $n(\beta_n^{(i)} - \beta_n^{(i)}) = -\frac{\pi}{2}$, for any number of full circumferences.

The conditions established by the data will therefore be satisfied.

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The variations observed of the earth's magnetization are therefore caused by a virtual magnetic field related to a primitive field by the following conditions:

"The modules of the virtual field are very small fractions of the modules of the instantaneous field, where the arguments differ from these by the angle $\frac{\pi}{2n}$."

On this assumption, the average yearly motion $m_n^{(i)}$ of any given term is equal to: $-\frac{N_i \alpha_n^{(i)}}{n\alpha_n^{(i)}}$ and the

corresponding magnetic moment's time of rotation about the earth's axis will be $\frac{2\pi n \alpha_n^{(i)}}{2\pi n \alpha_n^{(i)}}$.

If we let $m_n^{(i)}$ be the yearly motion of $\beta_n^{(i)}$, we may set:

$$\beta_n^{(i)} = m_n^{(i)} t + \gamma_n^{(i)};$$

$\gamma_n^{(i)}$ is the initial longitude which corresponds to the origin of time.

13.

Following these preliminary considerations, we will put the question in specific terms and we will ask ourselves which forces are capable of exerting the required virtual magnetization, for each instant?

It is not our intention to dispute the countless hypotheses put forth in this regard. We simply wish to propose what we consider to be the most likely hypothesis.

In our opinion, the phenomenon of the secular variations of the earth's magnetization is a large phenomenon of electromagnetic induction taking place in the rarified layers of the atmosphere, set in motion by the earth's rotation. Below, we will see that this hypothesis effectively accounts for the essential traits of the phenomenon.

The existence of induction currents in the atmosphere seems well established by other circumstances.

The role of the terrestrial atmosphere in the diurnal oscillations of the magnetic elements has been well-known by scientists since Schuster's report on this subject

which he had inserted in volume CLXXX of the Philosophical Transactions in London. The result is that this variation corresponds to a term of form $Y_1^{(2)}$ of the force function, where the argument increases uniformly with time and describes a full circumference in 24 hours. This term seems to be sort of related to the diurnal oscillations of the barometer.

I have demonstrated myself, in an unpublished note, that the semi-diurnal lunar variation of the terrestrial magnetic field is caused by electrical currents circulating in high regions of the atmosphere. The corresponding term in V is the expression $Y_2^{(3)}$; I demonstrated next that an atmospheric tidal wave would generate a term of this form.

In the cases mentioned, it is due to periodic motions /33 that the atmosphere becomes the center of induction currents under the action of the terrestrial magnet. Moreover, information is not lacking that there is a uniform and rotary motion of the high layers of the atmosphere, which seems to play an important role in the phenomenon of polar light.

Most likely, there is nothing *à priori* in this supposition that the rarified layers of the atmosphere become the center of electrical currents, capable of reacting to the earth itself, and which thus produce this continuous remagnetization of the earth which we observe the effects of.

14.

This brings us to study the currents induced in a homogeneous spherical layer whose particles are animated in any given rotary or oscillating motions.

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First, it is easy to see that the oscillating motions having a speed potential will generate a system of induction currents in such a manner that the current function will be of a higher order than that of the term that generates it and it will never have the same expression, as required by the observations.

This is no longer the case for the rotary motions of the atmosphere. Consequently, we will not discuss the problem in all its generalities, but will examine the hypothesis which seems to fit the facts the best, and this is the case of a spherical layer undergoing a uniform rotation about a stationary axis.

Moreover, this problem was the subject matter of Herz' first work (Oeuvres de Herz, t. I), so that we may benefit from the results of his analysis.

Let us consider a thin spherical layer, of radius R' undergoing a uniform rotary motion, of angular velocity ω , about the axis of a concentric spherical magnet. Each term

$$\left(\frac{R}{R'}\right)^{l+1} \cdot P_n^{(l)} \cdot \alpha_n^{(l)} \cos(n\varpi + n\beta_n^{(l)})$$

of the function of the magnetic forces, generates a similar term in the running function ϕ which has the following expression:

$$-\frac{R'}{i} \frac{\omega}{z} \left(\frac{R}{R'}\right)^{l+1} \cdot P_n^{(l)} \cdot n \alpha_n^{(l)} \sin(n\varpi + n\beta_n^{(l)}),$$

where χ represents the specific resistance of the surface.*

*Remember that we have not defined the magnetic potential $V = - \int \frac{dm}{r}$ contrary to most modern authors who have adopted the notation $V = \int \frac{dm}{r}$.

This system of induced currents itself generates a magnetic field, within the layer, and under our hypothesis, this magnetic field is identical to that V' discussed above.

Let us evaluate this field. By borrowing Maxwell's formula in his Treatise (volume II, art. 672), we find, for an inside point of distance r from the center of the sphere,

$$-4\pi \frac{i+1}{2i+1} \frac{R'}{r} \frac{\omega}{2} \left(\frac{r}{R'}\right)^i \left(\frac{R}{R'}\right)^{i+1} \cdot P_n^{(i)} \cdot n \alpha_n^{(i)} \sin(n\pi + n\beta_n^{(i)}).$$

This is indeed the expression of the required form. The hypothesis is therefore completely verified.

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By expressing the expression for the inductor field in its simplest form:

$$P_n^{(i)} \alpha_n^{(i)} \cos(n\pi + n\beta_n^{(i)}),$$

we find, for argument $\beta_n^{(i)}$, the following relationship $n \cdot \gamma_n^{(i)} = n\beta_n^{(i)} \pm \frac{\pi}{2}$. If ω is negative*, this will bring us to: $\cos(n\pi + n\beta_n^{(i)}) = + \sin(n\pi + n\beta_n^{(i)})$, and we must therefore take the lower sign and set:

$$\beta_n^{(i)} = \beta_n^{(i)} - \frac{\pi}{2n}.$$

Accordingly, if the relative motion of the upper layers of the atmosphere is directed in the opposite direction to the diurnal motion, the motions observed of $\beta_n^{(i)}$ coincide with the theory.

For the module $\alpha_n^{(i)}$, we have the following expression:

* ω is positively accounted for in the direction of increasing angular velocities.

$$\alpha_n^{(i)} = 4\pi \frac{i+1}{2i+1} \frac{R}{r} \left(\frac{r}{R}\right)^i \left(\frac{R}{r}\right)^{i+1} n \alpha_n^{(i)}.$$

Above, we found that (no. 12) the yearly motion of $\beta_n^{(i)}$ is equal to $\frac{N_i \alpha_n^{(i)}}{n \alpha_n^{(i)}}$: for functions of different orders, while the yearly motion is dependent upon both the order i and N_i , N_i being a complication function of i and on the law of variation of the induction coefficient within the earth. In our case, the factor N_i is essentially independent from i ; the yearly motion should be proportional, for the terms of different orders, to $\frac{i+1}{(2i+1)i}$, and therefore, specifically, for the functions of the first four orders, it would be approximately within the ratio of the fractions $\frac{2}{3.1}$, $\frac{3}{5.2}$, $\frac{4}{7.3}$, $\frac{5}{9.4}$. This condition is approximately fulfilled, with one important exception, by the motions which the observations enable us to establish with some certainty.

15.

The preceding analysis accounts for the main part of the changes in the earth's magnetization, and nothing proves that these are exact laws of the phenomenon. Conversely, it is probable that the changes are actually more complex, and it would be useful to become aware of what we might expect.

The formulas which we have given are appropriate for the case of a homogeneous sphere or a sphere composed of concentric or homogeneous layers. A heterogeneity of the magnetized sphere introduces periodical terms in $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ /35

in the solution.

Let us quickly examine what happens in a heterogeneous sphere which deviates very little from homogeneity. As a first approximation, we may still assume the $\alpha_n^{(i)}$ s are constant and the $\beta_n^{(i)}$ s are in the form $\gamma_n^{(i)} + m_n^{(i)}t$.

The differential equation which each $Y_n^{(i)}$ term of the function of forces satisfies, will still be:

$$\frac{dY_n^{(i)}}{dt} = Z_n^{(i)},$$

where $Z_n^{(i)}$ is expressed:

$$Z_n^{(i)} = N_i \bar{\alpha}_n^{(i)} \cos n(\varpi + \bar{\beta}_n^{(i)}).$$

However, here, $\bar{\alpha}_n^{(i)}$ and $\bar{\beta}_n^{(i)}$ will contain the periodical terms in $2n\beta_n^{(i)}$, or, which amounts to the same thing, in $2nm_n^{(i)}t$, $m_n^{(i)}$ being the yearly motion of $\beta_n^{(i)}$.

By expanding $Y_n^{(i)}$ and $Z_n^{(i)}$ and identifying the $\cos n\varpi$ and $\sin n\varpi$ coefficients of the two members, we will again have, by virtue of a known theorem:

$$\frac{d\bar{\alpha}_n^{(i)}}{dt} \cos n\bar{\beta}_n^{(i)} - n m_n^{(i)} \bar{\alpha}_n^{(i)} \sin n\bar{\beta}_n^{(i)} = N_i \bar{\alpha}_n^{(i)} \cos n\bar{\beta}_n^{(i)},$$

$$\frac{d\bar{\beta}_n^{(i)}}{dt} \sin n\bar{\beta}_n^{(i)} + n m_n^{(i)} \bar{\beta}_n^{(i)} \cos n\bar{\beta}_n^{(i)} = N_i \bar{\beta}_n^{(i)} \sin n\bar{\beta}_n^{(i)};$$

from which we may easily deduce:

$$\frac{d\bar{\alpha}_n^{(i)}}{dt} = N_i \bar{\alpha}_n^{(i)} \cos n(\bar{\beta}_n^{(i)} - \bar{\beta}_n^{(i)}); \quad \frac{d\bar{\beta}_n^{(i)}}{dt} = \frac{N_i \bar{\alpha}_n^{(i)}}{n\bar{\alpha}_n^{(i)}} \sin n(\bar{\beta}_n^{(i)} - \bar{\beta}_n^{(i)}).$$

In these equations $\bar{\alpha}_n^{(i)}$ and $\bar{\beta}_n^{(i)}$, and consequently $\bar{\alpha}_n^{(i)}$ and $\bar{\beta}_n^{(i)}$ have received small periodical increases $\delta\alpha_n^{(i)}$, $\delta\beta_n^{(i)}$, etc. which we are trying to determine.

By deducing the equations, which are analogous for the case studied earlier, from these equations, we will have the equations with the following differences:

$$\frac{1}{\alpha_n^{(i)}} \frac{d\alpha_n^{(i)}}{dt} = -N_i \left(\frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} + \delta \frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} \right) \sin n (\delta \beta_n^{(i)} - \delta \beta_n^{(i)}),$$

$$\frac{d\delta \beta_n^{(i)}}{dt} = \frac{N_i}{n} \left(\frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} + \delta \frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} \right) \cos n (\delta \beta_n^{(i)} - \delta \beta_n^{(i)}) - \frac{N_i}{n} \frac{\alpha_n^{(i)}}{\alpha_n^{(i)}}.$$

If the sphere under consideration does not deviate much from homogeneity, the angle $n(\beta_n^{(i)} - \beta_n^{(i)})$ will be close to $-\frac{\pi}{2}$ and the ratio $\frac{\alpha_n^{(i)}}{\alpha_n^{(i)}}$ will differ very little from $\frac{\alpha_n^{(i)}}{\alpha_n^{(i)}}$. By expanding the second members according to the powers of very small quantities σ and τ , defined by the relationships $\sigma = n(\beta_n^{(i)} - \beta_n^{(i)}) + \frac{\pi}{2}$ and $\frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} = \frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} (1 + \tau)$, and by omitting the fourth order quantities, we will have, by inverting the order of the differentiations in the first members:

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$$\delta \frac{d\alpha_n^{(i)}}{dt} = -N_i \alpha_n^{(i)} (\sigma + \sigma\tau - \frac{\sigma^3}{6}), \quad \delta \frac{d\beta_n^{(i)}}{dt} = \frac{N_i}{n} \frac{\alpha_n^{(i)}}{\alpha_n^{(i)}} (\tau - \frac{\sigma^2}{2} - \frac{\sigma^2\tau}{2}).$$

Without going to the bottom of the question for the time being, we can predict that the quantities σ and τ will be periodical with respect to $2n\beta_n^{(i)}$, for reasons of symmetry, and consequently with respect to $2nm_n^{(i)}t$. We therefore have:

$$\delta \frac{d\alpha_n^{(i)}}{dt} \text{ and } \delta \frac{d\beta_n^{(i)}}{dt} = (\text{Periodic terms in } 2nm_n^{(i)}t).$$

These equations will cause the $\alpha_n^{(i)}$'s and $\beta_n^{(i)}$'s to become integrated with the trigonometric series proceeding according to the multiples of the angle $2nm_n^{(i)}t$. The periodic time of the first term of the series will be $\frac{2\pi}{2nm_n^{(i)}}(i)$.

We can easily see that by assuming $\delta \frac{\alpha_n^{(i)}}{\alpha_n^{(j)}}$ and $n(\delta \beta_n^{(i)} - \delta \beta_n^{(j)})$ are of the same infinitesimal order, $\delta \alpha_n^{(i)}$ will be of the order of $\delta \beta_n^{(i)}$. This will occur, for example, if the heterogeneous sphere is assimilated with a homogeneous sphere surrounded by a circular ring passing through its poles. Thus, the periodic terms in $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ will be of the same order.

By closing this superficial discussion here, we may summarize the results as follows:

"In a heterogeneous sphere which substantially approaches a homogeneous sphere, the heterogeneity introduces periodical terms into the expressions of the $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ whose cycles will be the half-periodical times of the corresponding term in the function of forces, and its multiples".

These are the main special characteristics which we might hope to see come out of a more thorough discussion of the overall observations made, if the ideas put forth in this report are accurate.